Constantin Dumitrescu ■ Florentin Smarandache

METHODS OF SOLVING CALCULUS PROBLEMS

The Educational Publisher

Columbus, 2015

C. Dumitrescu F. Smarandache Methods of Solving Calculus Problems

Theory and Exercises

First imprint: *Metode de calcul în analiza matematică*
(in Romanian), INFOMED Publishing House,
Craiova, Romania, 1995

Translation to English by ANA MARIA BUZOIANU

The Educational Publisher
Zip Publishing
1313 Chesapeake Ave.
Columbus, Ohio 43212, USA
Email: info@edupublisher.com

Editura Ferestre
13 Cantemir str.
410473 Oradea, Romania

ISBN 978-1-59973-355-5

■ Constantin Dumitrescu ■ Florentin Smarandache ■

Methods of Solving Calculus Problems

Theory and Exercises

The Educational Publisher
Columbus, 2015

Motto:

Love is the purpose, the sacred command of existence.
Driven by it, eagles search each other in spaces,
Female dolphins brace the sea looking for their grooms
Even the stars in the skies cluster in constellations.
(Vasile Voiculescu)

Contents

Foreword

We rejoice expressing our gratitude towards all those who, better or less known, along the years, have helped us to get to this particular book. And they are many, many indeed whom have helped us! Some provided suggestions, others, ideas; at times, we have struggled together to decipher a certain elusive detail; other times we learned from the perceptive or sometime naïve questions of our interlocutors.

The coming into being of this book represents a very good example of altruism, kindness and selflessly giving, of the journey, made together, on the road to discovering life's beauty.

He who thinks not only of his mother or father, of himself or his family, will discover an even bigger family, will discover life everywhere, which he will start to love more and more in all its manifestations.

Mathematics can help us in our journey, stimulating and enriching not just our mental capacity, our reasoning, logic and the decision algorithms, but also contributing in many ways to our spiritual betterment. It helps shed light in places we considered, at first, fit only for an empirical evolution – *our own selves*.

We offer this succession of methods encountered in the study of high school calculus to students and teachers, to higher education entry examination candidates, to all those interested, in order to allow them to reduce as many diverse problems as possible to already known work schemes.

We tried to present in a methodical manner, advantageous for the reader, the most frequent calculation methods encountered in the study of the calculus at this level.

In this book, one can find:

- methods for proving the equality of sets,
- methods for proving the bijectivity of functions,
- methods for the study of the monotonic sequences and functions,
- mutual methods for the calculation of the limits of sequences and functions,
- specific methods for the calculation of the limits of sequences,
- methods for the study of continuity and differentiation,
- methods to determine the existence of an equation's root,
- applications of Fermat's, Rolle's, Lagrange's and Cauchy's theorems,
- methods for proving equalities and inequalities,
- methods to show that a function has primitives,
- methods to show that a function does not have primitives,
- methods to show that a function is integrable,
- methods to show that a function is not integrable.

We welcome your suggestions and observations for the improvement of this presentation.

C. Dumitrescu, F. Smarandache

I. Sets Theory

A set is determined with the help of one or more properties that we demand of its elements to fulfill.

Using this definition might trick us into considering that any totality of objects constitutes a set. Nevertheless it is not so.

If we imagine, against all reason, that any totality of objects is a set, then the totality of sets would form, in its own turn, a set that we can, for example, note with M. Then the family $\rho(M)$ of its parts would form a set. We would thus have $\rho(M) \in M$.

Noting with *card*M the number of elements belonging to M, we will have:

$$card\,\rho(M) \leq cardM.$$

However, a theorem owed to Cantor shows that we always have

$$card\,M < card\,\rho(M).$$

Therefore, surprisingly maybe, **not any totality of objects can be considered a set**.

Operations with sets

DEFINITION: The set of mathematical objects that we work with at some point is called a total set, notated with T.

For example,

- drawing sets on a sheet of paper in the notebook, the total set is the sheet of paper;
- drawing sets on the blackboard, the total set is the set of all the points on the blackboard.

It follows that **the total set is not unique**, it depends on the type of mathematical objects that we work with at some given point in time.

In the following diagrams we will represent the total set using a rectangle, and the subsets of T by the inner surfaces of this rectangle. This sort of diagram is called an **Euler-Venn diagram**.

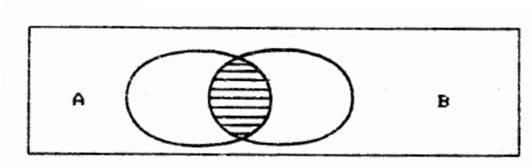
We take into consideration the following operations with sets:

1. The Intersection

$$A \cap B = \{x \in T \mid x \in A \; and \; x \in B\}$$

Because at a given point we work only with elements belonging to T, the condition $x \in T$ is already implied, so we can write:

$$A \cap B = \{x \mid x \in A \; and \; x \in B\}$$



A* ∩ *B

More generally,

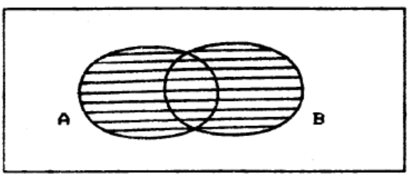
$$\bigcap_{i=1}^{\infty} A_i = \{ x \mid \forall i \in \mathbb{N}, \; x_i \in A \}$$

Let’s observe that:

$$x \notin A \cap B <===> x \notin A \; or \; x \notin B$$

2. The Union

$$A \cup B = \{x \mid x \in A \; orx \in B\}$$



A* ∪ *B

As in the case of the intersection, we can consider:

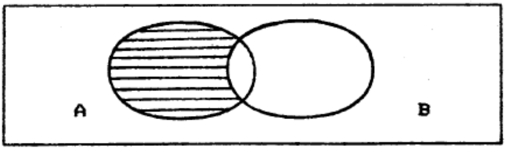
$$\bigcup_{i=1}^{\infty} A_i = \{ x \mid \exists\, i \in \mathbb{N} ,\ x_i \in A \}$$

We see that:

$$x \notin A \cup B <===> x \notin A \; and \; x \notin B$$

3. The Difference

$$A - B = \{x \mid x \in A \; and \; x \notin B\}$$



$A - B = C_A B$

We retain that:

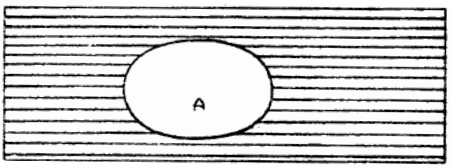
$$x \notin A - B <===> x \notin A \; or \; x \in B$$

4. The Complement

The complement of a set A is the difference between the total set and A.

$$C_T A = \{ x \mid x \in T \; and \; x \notin A\} = \{ x \mid x \notin A\}$$

The complement of a set is noted with CA or with $\bar{A}$.



$\boldsymbol{CA}$

Let's observe that

$$x \notin CA <===> x \in A$$

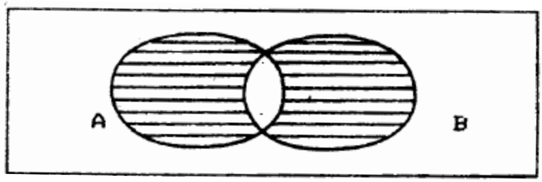
More generally, we can talk about the complement of a set to another, random set. Thus,

$$C_A B = \{\, x \mid x \in B \text{ and } x \notin A\}$$

is the complement of set B to set A.

5. The Symmetrical Difference

$$A \Delta B = (A - B) \cup (B - A)$$



$\boldsymbol{A \Delta B}$

We have $x \notin A\Delta B <===> x \notin A - B$ and $x \notin B - A$.

6. The Cartesian Product

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

The Cartesian product of two sets is a set of an **ordered pairs** of elements, the first element belonging to the first set and the second element belonging to the second set.

For example, $\mathbb{R} \times \mathbb{R} = \{(x, y) | x \in \mathbb{R}, y \in \mathbb{R}\}$, and an intuitive representation of this set is provided in *Figure 1.1*.

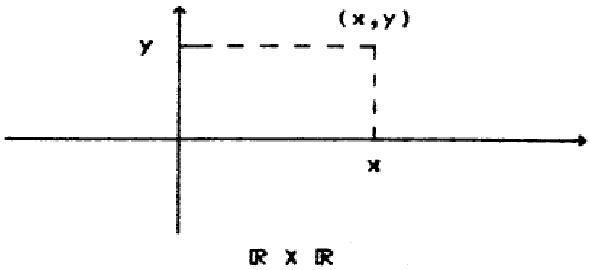
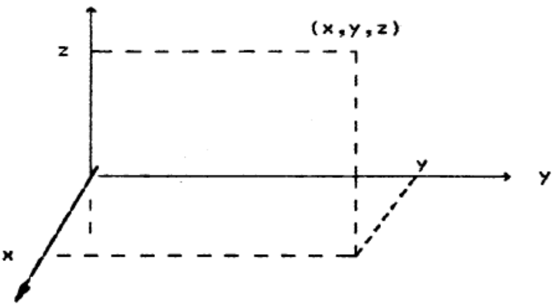


Fig. 1.1

By way of analogy, the Cartesian product of three sets is **a set of triplets**:

$$A \times B \times C = \{(x, y, z) | x \in A , y \in B, z \in C\}.$$

An intuitive image of $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is given in the figure below.



ℝ X ℝ X ℝ

Fig. 1.2

More generally,

$$\mathop{X}_{i=1}^{n} A_i = \{(X_1, X_2, \ldots, X_n) | \ X_i \ \in A_i \ for \ any \ i \in \overline{1, n}\}.$$

Methods for Proving the Equality of Two Sets

The difficulties and surprises met on the process of trying to rigorously define the notion of sets are not the only ones we encounter in the theory of sets.

Another surprise is that the well-known (and the obvious, by intuition) method to prove the equality of two sets using the **double inclusion is** – at the level of a rigorous definition of sets – **just an axiom.**

$$A = B <===> A \subseteq B \ and \ B \subseteq A \qquad (1.1)$$

By accepting this axiom, we will further exemplify two methods to prove the equality of sets:

(A) THE DOUBLE INCLUSION [expressed through the equivalence (1.1)];

(B) UTILIZING THE CHARACTERISTIC FUNCTION OF A SET.

We will first provide a few details of this second method that is much faster in practice, hence more convenient to use than the first method.

DEFINITION. We call a characteristic function of the set A the function $\varphi_A : T \longrightarrow \{0,1\}$ defined through:

$$\varphi_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A \end{cases}$$

As we can observe, the numbers 0 and 1 are used to divide the elements of the total set T in two categories:

(1) one category contains those elements x in which the value of φ_A is 1 (the elements that belong to A);

(2) all the elements in which the value of φ_A is 0 (elements that do not belong to A) belong to the second category.

Method (B) of proving the equality of two sets is based on the fact that any set is uniquely determined by its characteristic function, in the sense that: **there exists a bijection from set$\boldsymbol{\varphi(T)}$ of the subsets of $\boldsymbol{T}$ to the set $\boldsymbol{\gamma(T)}$ of the characteristic functions defined on $\boldsymbol{T}$** (see *Exercise VII*).

Hence,

$$A = B <===> \varphi_A = \varphi_B. \quad (1.2)$$

Properties of the Characteristic Function

$\varphi_1)$ $\varphi_{A\cap B}(x) = \varphi_A(x).\varphi_B(x)$

$\varphi_2)$ $\varphi_{A\cup B}(x) = \varphi_A(x) + \varphi_B(x) - \varphi_{A\cap B}(x)$

$\varphi_3)$ $\varphi_{A-B}(x) = \varphi_A(x) - \varphi_{A\cap B}(x)$

$\varphi_4)$ $\varphi_{CA}(x) = 1 - \varphi_A(x)$

φ_5) $\varphi_{A\Delta B}(x) = \varphi_A(x) + \varphi_B(x) - 2.\varphi_{A\cap B}(x)$
φ_6) $\varphi_{A\times B}(x,y) = \varphi_A(x) \times \varphi_B(y)$
φ_7) $\varphi_A^2(x) = \varphi_A(x)$
φ_8) $\varphi_\emptyset(x) \equiv 0,\ \varphi_T(x) \equiv 1$
φ_9) $A \subseteq B <===> \varphi_A(x) \leq \varphi_B(x)$ for any $x \in T$.

Let's prove, for example φ_1). For this, let's observe that the total set T is divided by sets A and B in four regions, at most:

1) for the points x that do belong to neither A, nor B, we have $\varphi_A(x) = \varphi_B(x) = 0$ and $\varphi_{A\cap B}(x) = 0$.

2) for the points x that are in A but are not in B, we have $\varphi_A(x) = 1, \varphi_B(x) = 0\ and\ \varphi_{A\cap B}(x) = 0$.

3) if $x \notin A\ and\ x \in B$,

the equalities follow analogously.

4) if $x \in B\ and\ x \notin A$

Exercises

I. Using methods (A) and (B) show that:

1. $A - (B \cap C) = (A - B) \cup (A - C)$
2. $A - (B \cup C) = (A - B) \cap (A - C)$
3. $C(B \cap C) = CB \cup CC$
4. $C(B \cup C) = CB \cap CC$
5. $A \cap (B - C) = (A \cap B) - (A \cap C)$
6. $A \cup (B - A) = A \cup B$
7. $A - (A - B) = A \cap B$
8. $C(CB) = B$
9. $A \cup CA = T$, $A \cap CA = \emptyset$

SOLUTIONS:

(A) (the double inclusion)

Let's notice that solving a mathematical problem requires we successively answer two groups of questions:

(q) WHAT DO WE HAVE TO PROVE?

(Q) HOW DO WE PROVE?

Answering question (Q) leads us to a new (q) question. Thus, for exercise 1, the answer to the question (q) is:

(r_1): We have to prove an equality of sets,

and the answer to the question (Q) is:

(R_1): We can prove this equality using two methods: using the double inclusions or using the properties of the characteristic function.

We choose the first method and end up again at (q). This time the answer is:

(r_2): We have to prove two inclusions,

and the answer to the corresponding question (Q) is:

(R_2): We prove each inclusion, one at a time.

Then,

(r_3): We have to prove one inclusion;

(R_3): We prove that an arbitrary element belonging to the "included set" belongs to the "set that includes".

It is obvious that this entire reasoning is mental, the solution starts with:

a) Let $x \in A-(B-C)$, irrespective [*to continue we will read the final operation (the difference)*], $<===> x \in A$ and $x \notin B \cap C$ [*we will read the operation that became the last (the union)]* $<===>$

$$x \in A \text{ and} \begin{cases} x \notin B \\ \quad or \\ x \notin C \end{cases} <===> \begin{cases} x \in A \; and \; x \notin B \\ \qquad or \\ x \in A \; and \; x \notin C \end{cases} \qquad (1.3)$$

From this point on we have no more operations to explain and we can regroup the enunciation we have reached in several ways (many implications flow from (1.3), but we are only interested in one - the conclusion of the exercise), that is why we have to update the conclusion:

$$x \in (A-B) \cup (A-C) <===> x \in A-B \; or \; x \in A-C.$$

We observe from (1.3) that this conclusion follows immediately, so the inclusion is proven.

For the second inclusion: Let $x \in (A-B) \cup (A-C)$, [*we will read the final operation (the reunion)*] <===>
$\begin{cases} x \in A-B \\ \quad or \\ x \in A\ -C \end{cases}$ <===> [*we will read the operation that became the last*]
<===>

$$\begin{cases} x \in A \; and \; x \notin B \\ \qquad or \\ x \in A \; and \; x \notin C \end{cases} \quad <===> \quad x \in A \text{ and} \begin{cases} x \notin B \\ or \\ x \notin C \end{cases} \qquad (1.4)$$

We have no more operations to explain, so we will describe in detail the conclusion we want to reach: $x \in A-(B \cap C)$, i.e. $x \in A$ and $x \notin B \cap C$, an affirmation that immediately issues from (1.4).

5. (r_1): We have to prove an equality of sets;

(R_1): We prove two inclusions;

(r_2): We have to prove an inclusion;

(R_2): We prove that each element belonging to the included set is in the set that includes.

Let $x \in A-(A-B)$, irrespective [we will read the final operation (the difference)], $<===>$ $x \in A$ and $x \notin A-B$ [*we will read the operation that became the last*] $<===>$

$$x \in A \text{ and} \begin{cases} x \notin A \\ or \\ x \in B. \end{cases} \qquad (1.5)$$

We have no more operations to explain, so we update the conclusion:

$x \in A \cap B \; i.e. x \in A \; and \; x \in B.$

We observe that (1.5) is equivalent to:

$$\begin{cases} x \in A \; and \; x \notin A \\ \qquad or \\ x \in A \; and \; x \in B. \end{cases}$$

The first affirmation is false, and the second one proves that$x \in A \cap B$.

We will solve the same exercises using the properties of the characteristic function and the equivalence (1.2).

1. We have to prove that $\varphi_{A-(B\cap C)} = \varphi_{(A-B)\cup(A-C)}$;

(r_1): We have to prove an equality of functions;

(R_1): As the domain and codomain of the two functions coincide, we have to further prove that their values coincide, i.e.

(r_2): $\varphi_{A-(B\cap C)}(x) = \varphi_{(A-B)\cup(A-C)}(x)$ for any $x \in T$.

The answer to the corresponding (Q) question is

(R_2): We explain both members of the previous equality.

$\varphi_{A-(B\cap C)}(x) =$ [we will read the final operation (the difference) and we will apply property φ_3] $= \varphi_A(x) - \varphi_{A\cap(B\cap C)}(x) =$ [we will read the operation that became the last and we will apply property$\boldsymbol{\varphi_1}$] =

$$\varphi_A(x) - \varphi_A(x).\varphi_B(x).\varphi_C(x) \qquad (1.6)$$

The second member of the equality becomes successive:

$$\varphi_{(A-B)\cup(A-C)}(x) = \varphi_{A-B}(x) + \varphi_{A-C}(x) - \varphi_{(A-B)\cap(A-C)}(x) =$$
$$\left(\varphi_A(x) - \varphi_{A\cap B}(x)\right) + \left(\varphi_A(x) - \varphi_{A\cap C}(x)\right) -$$
$$- \varphi_{A-B}(x)\cdot\varphi_{A-C}(x) = \left(\varphi_A(x) - \varphi_A(x)\cdot\varphi_B(x)\right) +$$
$$+ \left(\varphi_A(x) - \varphi_A(x)\cdot\varphi_C(x)\right) - \left(\varphi_A(x) - \varphi_{A\cap B}(x)\right).$$
$$\cdot\left(\varphi_A(x) - \varphi_{A\cap C}(x)\right) = 2.\varphi_A(x) - \varphi_A(x).\varphi_B(x) -$$
$$- \varphi_A(x).\varphi_C(x) - \varphi_A^2(x) + \varphi_A(x)\cdot\varphi_{A\cap C}(x) +$$
$$+ \varphi_A(x)\cdot\varphi_{A\cap B}(x) - \varphi_A^2(x).\varphi_B(x).\varphi_C(x) =$$
$$= \varphi_A(x)\cdot\varphi_B(x)\cdot\varphi_C(x).$$

5. In order to prove the equality $\varphi_{A-(A-B)}(x) - \varphi_{A\cap B}(x)$ we notice that:

$$\varphi_{A-(A-B)}(x) = \varphi_A(x) - \varphi_{A\cap(A-B)}(x) = \varphi_A(x) -$$
$$- \varphi_A(x)\cdot\varphi_{A\cap B}(x) = \varphi_A(x) - \varphi_A(x)\cdot$$

$$\cdot\left(\varphi_A(x) - \varphi_{A\cap B}(x)\right) = \varphi_A(x) - \varphi_A(x)\cdot\left(\varphi_A(x) - \right.$$
$$\left. - \varphi_A(x)\cdot\varphi_B(x)\right) = \varphi_A(x)\cdot\varphi_B(x) = \varphi_{A\cap B}(x).$$

II. Prove that:

1. $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times C)$
2. $(A \cup B) \times C = (A \times C) \cup (B \times C)$
3. $(A - B) \times C = (A \times C) - (B \times C)$
4. $A \Delta (B \Delta C) = (A \Delta B) \Delta C$
5. $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$
6. $(A \Delta B) \times C = (A \times C) \Delta (B \times C)$

SOLUTIONS.

Method (A) (the double inclusion)

1. Let $x \in (A \cap B) \times (C \cap D)$, irrespective [*we explain the final operation (the scalar product)*] $<===>$ $x = (\alpha, \beta)$, with $\alpha \in A \cap B$ and $\beta \in C \cap D$ $<===>$[*we explain the operations that became the last ones*] $<===>$ $x = (\alpha, \beta)$ with $\alpha \in A$, $\alpha \in B$ and $\beta \in C$, $\beta \in D$ [*we have no more operations to explain so we update the conclusion*: $x \in (A \times C) \cap (B \times D)$, *so* x *belongs to an intersection (the last operation)*] $<===>$ $x = (\alpha, \beta)$ with $\alpha \in A$, $\beta \in C$ and $\alpha \in B$, $\beta \in D$ $<===>$ $x \in A \times C$ and $x \in B \times D$.

3. Let $x \in (A - B) \times C$, irrespective $<===>$ $x \in (\alpha, \beta)$ with $\alpha \in A - B$ and $\beta \in C$ $<===>$ $x \in (\alpha, \beta)$ with $\alpha \in A$ and $\alpha \notin B$ and $\beta \in C$ [*by detailing the conclusion we deduce the following steps*] $<===>$ $x = (\alpha, \beta)$, $\alpha \in A$, $\beta \in C$ and $x = (\alpha, \beta)$, $\alpha \notin B$, $\beta \in C$ $===>$ $x \in A \times C$ and $x \notin B \times C$.

Reciprocally, let $x \in (A \times C) - (B \times C)$, irrespective $<===>$ $x = (\alpha, \beta)$ with $(\alpha, \beta) \in A \times C$ and $\times C$ $<===>$ $x = (\alpha, \beta), \alpha \in A$ and $\beta \in C$ and

$$\begin{cases} \alpha \notin B \\ or \\ \beta \notin C \end{cases} <===> \begin{cases} \alpha \in A \; and \; \beta \in C \; and \; \alpha \notin B \\ or \\ \alpha \in A \; and \; \beta \in C \; and \; \beta \notin C \;\; (fals) \end{cases}$$

$===> (\alpha, \beta) \in (A - B) \times C.$

Method (B) (using the characteristic function)

$$1. \varphi_{(A \cap B) \times (C \cap D)}(x,y) = \varphi_{A \cap B}(x) \cdot \varphi_{C \cap D}(y) =$$
$$= \varphi_A(x) \cdot \varphi_B(x) \cdot \varphi_C(y) \cdot \varphi_D(y) \; \varphi_{(A \times C) \cap (B \times D)}(x,y) =$$
$$= \varphi_{A \times C}(x,y) \cdot \varphi_{B \times D}(x,y) = \varphi_A(x) \cdot \varphi_C(y) \cdot \varphi_B(x) \cdot \varphi_D(y)$$

$$3. \quad \varphi_{(A-B) \times C}(x,y) = \varphi_{A-B}(x) \cdot \varphi_C(y) =$$
$$= \left(\varphi_A(x) - \varphi_{A \cap B}(x) \right) \varphi_C(y) = \varphi_A(x) \cdot \varphi_C(y) -$$
$$- \varphi_A(x) \cdot \varphi_B(x) \cdot \varphi_C(y) . \; \varphi_{(A \times C)-(B \times C)}(x,y) =$$
$$= \varphi_{A \times C}(x,y) - \varphi_{(A \times C) \cap (B \times C)}(x,y) = \varphi_{A \times C}(x,y) -$$
$$- \varphi_{A \times C}(x,y) \cdot \varphi_{B \times C}(x,y) = \varphi_A(x) \cdot \varphi_C(y) -$$
$$- \varphi_A(x) \cdot \varphi_B(x) \cdot \varphi_C^2(y) \; .$$

$$4. \quad \varphi_{A \Delta (B \Delta C)}(x) = \varphi_A(x) + \varphi_{B \Delta C}(x) -$$
$$- 2\varphi_A(x) \cdot \varphi_{B \Delta C}(x) = \varphi_A(x) + \varphi_B(x) + \varphi_C(x) -$$
$$- 2\varphi_B(x) \cdot \varphi_C(x) - 2\varphi_A(x) \left(\varphi_B(x) + \varphi_C(x) - \right.$$
$$\left. - 2\varphi_B(x) \cdot \varphi_C(x) \right) = \varphi_A(x) + \varphi_B(x) +$$
$$+ \varphi_C(x) - 2 \cdot \varphi_A(x) \cdot \varphi_B(x) - 2 \cdot \varphi_A(x) \cdot \varphi_C(x) -$$
$$- 2 \cdot \varphi_B(x) \cdot \varphi_C(x) + 4 \cdot \varphi_A(x) \cdot \varphi_B(x) \cdot \varphi_C(x) \; .$$

Explaining $\varphi_{(A\Delta B)\Delta C}(x)$, in an analogue manner, we obtain the desired equality. If we observe that the explanation of $\varphi_{(A\Delta B)\Delta C}(x)$ from above is symmetrical, by utilizing the commutativity of addition and multiplication, the required equality follows easily.

$$5.\ \varphi_{(A\Delta B)\times C}(x,y) = \varphi_{A\Delta B}(x)\cdot\varphi_C(y) =$$
$$=\left(\varphi_A(x) + \varphi_B(x) - 2\cdot\varphi_A(x)\beta_B(x)\right)\varphi_C(y);$$
$$;\varphi_{(A\times C)\Delta(B\times C)}(x,y) = \varphi_{A\times C}(x,y) + \varphi_{B\times C}(x,y) -$$
$$- 2\cdot\varphi_{A\times C}(x,y)\ .\ \varphi_{B\times C}(x,y) = \varphi_A(x)\cdot\varphi_C(y) +$$
$$+\varphi_B(x)\cdot\varphi_C(y) - 2\cdot\varphi_A(x)\cdot\varphi_B(x)\cdot\varphi_C^2(y).$$

Equalities 4)-5) were easy to prove using the characteristic function; they are, however, more difficult to prove using the first method. The characteristic function is usually preferred, due to the ease of use and the rapidity of reaching the result.

III. Prove the following equivalences:

1. $A \cup B \subset C \iff A \subset C$ and $B \subset C$
2. $A \subset B \cap C \iff A \subset B$ and $B \subset C$
3. $A \cap B \subset C \iff A \cap C_B \cup C$
4. $A \subset B \cup C \iff A \cap C_B \subset C$
5. $(A - B) \cup B \iff B \cup A$

SOLUTIONS:

1. r_1: We have to prove an equivalence;
   R_1: We prove two implications;
   r_2: $A \cup B \subset C \Rightarrow A \subset C\ and\ B \subset C$
   R_2: We prove that $A \subset C\ and\ B \subset C$ (two inclusions).

Let $x \in A$ irrespective [we have no more operations to explain, therefore we will update the conclusion: $x \in C$; but this is not evident from $x \in A$, but only with the help of the hypothesis]

$$x \in A \Rightarrow x \in A \cup B \Rightarrow [through\ the\ hypothethis\ A \cup B \subset C] \Rightarrow x \in C$$

Let $x \in B$ irrespective => $x \in A \cup B \subset C \Rightarrow x \in C$.

Reciprocally (the second implication):

r1: We have to prove that $A \cup B \subset C$ (an inclusion)

R1: Let $x \in A \cup B$ irrespective (we explain the last operation):

$$=> \begin{cases} x \in A => x \in C \ (due\ to\ the\ hypothesis) \\ \qquad\qquad\qquad or \\ x \in B => x \in C (due\ to\ the\ hypothesis) \end{cases}$$

5. r1: We have to prove an equivalence;

R1: We prove two implications;

r2: $(A - B) \cup B = A => B \subset A$;

R2: We prove the conclusion (an inclusion), using the hypothesis.

Let $x \in B$ irrespective => $x \in (A - B) \cup B => x \in A$

r3: We have to prove the implication: $B \subset A => (A - B) \cup B = A$;

R3: We prove an equality of sets (two inclusions).

For the first inclusion, let $x \in (A - B) \cup B$ irrespective [we explain the last operation] =>

$$=> \begin{cases} x \in A - B \\ \quad or \\ \quad x \in B \end{cases} => \begin{cases} x \in A\ and\ x \notin B & (a) \\ \qquad or & \\ x \in B & (b) \end{cases}$$

[We have no more operation to explain, so we update the conclusion]. We thus observe that from (a) it follows that $x \in A$, and from (b), with the help of the hypothesis, $B \subset A$, we also obtain $x \in A$.

Method 2:

$$1. \left(A \cup B \subset C \ => \ A \subset C \ \text{şi} \ B \subset C \right) <=>$$

$$<=> \left(\varphi_{A \cup B} < \varphi_C => \varphi_A < \varphi_C \ \text{şi} \ \varphi_B < \varphi_C \right)$$

We have to prove two inequalities among functions. Through the hypothesis, $\varphi_{A\cup B} < \varphi_C$ and, moreover $\varphi_A < \varphi_{A\cup B}$ (indeed, $\varphi_A < \varphi_{A\cup B} <=> \varphi_A < \varphi_A + \varphi_B - \varphi_A\varphi_B <=> \varphi_B(1 - \varphi_A) > 0$ – true, because the characteristic functions take two values: 0 and 1).

It follows that $\varphi_A < \varphi_{A\cup B} < \varphi_C$ and, analogously $\varphi_B < \varphi_{A\cup B} < \varphi_C$.

Reciprocally, for the inverse implication, we have to prove that: $\varphi_{A\cup B} < \varphi_C$, i.e.

$$\varphi_A(x) + \varphi_B(x) - \varphi_{A\cap B}(x) < \varphi_C(\text{x}) \qquad (1.7)$$

in the hypothesis:

$$\varphi_A < \varphi_C \text{ and } \varphi_B < \varphi_C \qquad (1.8)$$

But $\varphi_A, \varphi_B, \varphi_C$ can take only two values: 0 and 1. From the eight possible cases in which (1.7) must be checked, due to (1.8), there remain only four possibilities:

a) $\varphi_A(x) = \varphi_B(x) = \varphi_C(x) = 1$

b) $\varphi_A(x) = \varphi_C(x) = 1 \ , \ \varphi_B(x) = 0$

c) $\varphi_B(x) = \varphi_C(x) = 1 \ , \ \varphi_A(x) = 0$

d) $\varphi_A(x) = \varphi_B(x) = \varphi_C(x) = 0$

In each of these cases (1.7) checks out.

5. We have to prove that

$$\varphi_{(A-B)\cup B}(x) = \varphi_A(x) \quad \Leftrightarrow \quad \varphi_B(x) < \varphi_A(x)$$

But

$$\varphi_{(A-B)\cup B}(x) = \varphi_{A-B}(x) + \varphi_B(x) - \varphi_{(A-B)\cap B}(x) =$$
$$= \varphi_A(x) - \varphi_{A\cap B}(x) + \varphi_B(x) - \varphi_{A-B}(x)\cdot\varphi_B(x) =$$
$$= \varphi_A(x) - \varphi_A(x)\cdot\varphi_B(x) + \varphi_B(x) -$$
$$- \left(\varphi_A(x) - \varphi_A(x)\cdot\varphi_B(x) \right)\cdot\varphi_B(x) =$$
$$= \varphi_A(x) + \varphi_B(x) - \varphi_A(x)\cdot\varphi_B(x). \qquad (1.9)$$

For the direct implication, through the hypothesis, we have:

$$\varphi_A(x) + \varphi_B(x) - \varphi_A(x)\cdot\varphi_B(x) = \varphi_A(x), \text{ i.e.}:$$

$$\varphi_B(x)\left(1 - \varphi_A(x) \right) = 0 \ .$$

Consequently:

$$\begin{cases} \varphi_B(x) = 0 => \varphi_B(x) < \varphi_A(x) \ (\textit{due to the values } 0 \textit{ and } 1 \textit{ that the function } \varphi \textit{ takes}) \\ \qquad\qquad\qquad or \\ 1 - \varphi_A(x) = 0 => \varphi_A(x) = 1 \ so\ \varphi_B(x) < \varphi_A(x) \end{cases}$$

Reciprocally, if $\varphi_B(x) < \varphi_A(x)$, we cannot have $\varphi_B(x) = 1$ and $\varphi_A(x) = 0$, and from (1.9) we deduce:

$$\varphi_{(A-B)\cup B}(x) = \varphi_A(x) <=> \varphi_A(x) + \varphi_B(x) -$$
$$- \varphi_A(x) \cdot \varphi_B(x) = \varphi_A(x) <=> \varphi_B(x)\left(1 - \varphi_A(x)\right) = 0$$

- equality that is true for the three remaining cases:

a) $\varphi_A(x) = 1$, $\varphi_B(x) = 0$

b) $\varphi_A(x) = \varphi_B(x) = 1$

c) $\varphi_A(x) = \varphi_B(x) = 0$

IV. Using the properties of the characteristic function, solve the following equations and systems of equations with sets:

1. $A \cup (B - X) = B \cup X$

2. $\begin{cases} A - X = B \\ A \cup X = C \end{cases}$ if $B \subset A$ and $A \cap C = \emptyset$

3. $\begin{cases} A \cap X = B \\ A \cup X = C \end{cases}$ if $B \subset A \subset C$

4. $\begin{cases} X \cup Y = A \\ X \cap Y = B \\ X - Y = C \end{cases}$ if $B \subset A, C \subset A, B \cap C = \emptyset$

SOLUTION:

1. $A \cup (B - X) = B \cup X <=> \varphi_{A\cup(B-X)}(x) =$
$$= \varphi_{B\cup X}(x) <=> \varphi_A(x) + \varphi_{B-X}(x) - \varphi_A(x) \cdot \varphi_{B-X}(x) =$$
$$= \varphi_B(x) + \varphi_X(x) - \varphi_B(x) \cdot \varphi_X(x) <=> \varphi_A(x) +$$
$$+ \varphi_B(x) - \varphi_B(x) \cdot \varphi_X(x) - \varphi_A(x) \cdot \varphi_B(x) +$$

$$+\varphi_A(x)\cdot\varphi_B(x)\cdot\varphi_X(x) = \varphi_B(x) + \varphi_X(x) -$$
$$-\varphi_B(x)\cdot\varphi_X(x) \Leftrightarrow \varphi_X(x)\big(1 - \varphi_A(x)\cdot\varphi_B(x)\big) =$$
$$= \varphi_A(x)\big(1 - \varphi_B(x)\big) \Leftrightarrow \varphi_X(x)\cdot\varphi_{C(A\cap B)}(x) =$$
$$= \varphi_{A\cap CB}(x) \qquad (1.10)$$

We have to analyze the following cases:

a) if $x \notin A$ and $x \notin B$, we have:

$\varphi_{C(A\cap B)}(x) = 1$ and $\varphi_{A\cap CB}(x) = 1$, so for (1.10) to be accomplished we must have $\varphi_x(x) = 0$. So, any point that doesn't belong to A or B, doesn't belong to X either.

b) if $x \in A$ and $x \notin B$, we have:

$\varphi_{C(A\cap B)}(x) = 1$ and $\varphi_{A\cap CB}(x) = 1$, so for (1.10) to be accomplished we must have $\varphi_x(x) = 1$. Therefore, any point from $A - B$ is in X.

c) if $x \notin A$ and $x \in B$ and for (1.10) to be accomplished, we must have $\varphi_x(x) = 0$, consequently no point from $B - A$ is not in X.

d) if $x \in A$ and $x \in B$ we can have $\varphi_x(x) = 0$ or $\varphi_x(x) = 1$.

So any point from $A \cap B$ can or cannot be in X. Therefore $X = (A\backslash B) \cup D$, where $D \subset A \cap B$, arbitrarily.

2. $A - X = B \Leftrightarrow \varphi_{A-B}(x) = \varphi_B(x) \Leftrightarrow$

$$\Leftrightarrow \varphi_A(x) - \varphi_A(x)\cdot\varphi_X(x) = \varphi_B(x) \Leftrightarrow$$
$$\Leftrightarrow \varphi_X(x)\cdot\varphi_A(x) = \varphi_A(x) - \varphi_B(x) \qquad (1.11)$$
$$X - A = C \Leftrightarrow \varphi_X(x) - \varphi_X(x)\cdot\varphi_A(x) = \varphi_C(x) \Leftrightarrow$$
$$\Leftrightarrow \varphi_X(x)\big(1 - \varphi_A(x)\big) = \varphi_C(x) \Leftrightarrow$$
$$\Leftrightarrow \varphi_X(x)\cdot\varphi_{CA}(x) = \varphi_C(x) \qquad (1.12)$$

We have to analyze four cases, as can be seen from the following table:

Cases	$\varphi_A(x)$	$\varphi_B(x)$	$\varphi_C(x)$	$\varphi_X(x)$	(1.11)	(1.12)
$x \notin A$ and $x \notin C$	0	0	0	0	$0 \cdot 0 = 0 - 0$ (A)	$0 \cdot 1 = 0$ (A)
				1	$1 \cdot 0 = 0 - 0$ (A)	$1 \cdot 1 = 0$ (F)
$x \in A$ and $x \notin B$	1	0	0	0	$0 \cdot 1 = 1 - 0$ (F)	$0 \cdot 0 = 0$ (A)
				1	$1 \cdot 1 = 1 - 0$ (A)	$1 \cdot 0 = 0$ (A)
$x \in B$	1	1	0	0	$0 \cdot 1 = 1 - 1$ (A)	$0 \cdot 0 = 0$ (A)
				1	$1 \cdot 1 = 1 - 1$ (F)	$1 \cdot 0 = 0$ (A)
$x \in C$	0	0	1	0	$0 \cdot 0 = 0 - 0$ (A)	$0 \cdot 1 = 1$ (F)
				1	$1 \cdot 0 = 0 - 0$ (A)	$1 \cdot 1 = 1$ (A)

So $X = (A - B) \cup C$.

V. Determine the following sets:

1. $A = \{ x \in \mathbb{Z} \mid x = \dfrac{6n - 7}{2n + 1}, n \in \mathbb{Z} \}$

2. $A = \{ x \in \mathbb{N} \mid x = \dfrac{3n^2 - 2n + 1}{n^2 - 1}, n \in \mathbb{N} \}$

3. $A = \{ x \in \mathbb{Z} \mid x = \dfrac{6n^2 + 7}{3n + 1}, n \in \mathbb{Z} \}$

4. $A = \{ x \in \mathbb{Z} \mid \dfrac{x^3 - 3x + 2}{2x + 1} \in \mathbb{Z} \}$

5. $A = \{ (x,y) \in \mathbb{N} \times \mathbb{N} \mid 9y^2 - (x + 1)^2 = 32 \}$

6. $A = \{ (x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x^2 - 2xy + 3y^2 = 8 \}$

7. Let $P \in \mathbb{Z}(x)$ a n degree polynomial and $q \in \mathbb{Z}$. Knowing that $P(q) = 15$, determine the set:

$$A = \{ x \in \mathbb{Z} \mid \frac{P(x)}{x - q} \in \mathbb{Z} \}$$

8. $A = \{ x \in \mathbb{R} \mid \exists\, a \in \mathbb{R},\ x = \dfrac{a^2 - a + 1}{a + 1} \}$.

INDICATIONS:

1. An integer maximal part is highlighted. This is obtained making the divisions:

$$x = 3 - \frac{10}{2n + 1}, \text{ so } x \in \mathbb{Z} \iff \frac{10}{2n + 1} \in \mathbb{Z} \iff$$

10 is dividable by $2n+1$ $<=>$ $n \in \{0,1,-2\}$

$$4.\ E = \frac{x^2-3x+2}{2x+1} = \frac{1}{5}\left(4x^2-2x-11+\frac{27}{2x+1}\right),$$

so $E \in \mathbb{Z} <=> 27$ is dividable by $2x+1$ and $4x^2-2x-11+\frac{27}{2x+1}$ is a multiple of $8 <=> A \in \{-14,-5,-2,-1,0,1,4,13\}$.

$$5.\ 9y^2-(x+1)^2 = 32 <=> (3y-x-1)(3y+x+1) = 32$$

so x and y are integer solutions of systems the shape of:

$\begin{cases} 3y-x-1=u \\ 3y+x+1=v \end{cases}$ with u and v as divisors of 32.

$$6.\ x^2-2x+3y^2 = 8 <=> (x-y)^2+2y^2 = 8 =>$$

$x-y=0$ and $2y^2=8$.

$$7.\ P(x) = C(x)(x-q)+R <=> a_n x^n +$$

$$+ a_{n-1}x^{n-1} + \ldots + a_0 = C(x)(x-q)+15.$$

So $\frac{P(x)}{x-q} \in \mathbb{Z} <=> 15$ is dividable by $x-q$.

8. *Method 1*: For $a \neq -1$ we have

$$x = \frac{a^2-a+1}{a+1} <=> ax+x = a^2-a+1 <=>$$

$$<=> a^2-a(1+x)+1-x = 0 <=>$$

$$<=> a = \frac{1+x\pm\sqrt{x^2+6x-3}}{2}$$

so, we must have $x^2+6x-3>0$. As $x_{1,2} = -3\pm 2\sqrt{3}$, we deduce:

$$x \in (-\infty, -3-2\sqrt{3}] \cup [-3+2\sqrt{3}, +\infty).$$

Method 2: We consider the function $f(a) = \frac{a^2-a+1}{a+1}$ and $A = f(D)$, with $D = \mathbb{R}\{-1\}$ the domain of f. A is obtained from the variation table of f.

VI. Determine the following sets, when $a, b \in \mathbb{N}$:

$$1.\ A = \left\{ x \in \mathbb{Z} \,\middle|\, \left[\frac{a+x}{b}\right] = \left[\frac{a}{b}\right] \right\}$$

$$2.\ A = \left\{ x \in \mathbb{Z} \,\middle|\, \left[\frac{a+x}{b}\right] = \left[\frac{a}{b}\right] + 1 \right\}$$

$$3.\ A = \left\{ x \in \mathbb{R} \,\middle|\, \left[2^{x} \right] = 2^{[x]} \right\}$$

where $[t]$ is the integer part of t.

SOLUTIONS: We use the inequalities: $[t] < t < [t] + 1$, so:

$$1.\ \left[\frac{a+x}{b}\right] \leq \frac{a+x}{b} < \left[\frac{a+x}{b}\right] + 1$$

$$x \in A \iff \left[\frac{a}{b}\right] \leq \frac{a+x}{b} < \left[\frac{a}{b}\right] + 1 \iff$$

$$\iff b\left[\frac{a}{b}\right] - a \leq x < b\left[\frac{a}{b}\right] + b - a$$

$$2.\ x \in A \iff \left[\frac{a}{b}\right] + 1 \leq \frac{a+x}{b} < \left[\frac{a}{b}\right] + 2 \iff$$

$$\iff b\left[\frac{a}{b}\right] + b - a \leq x < b\left[\frac{a}{b}\right] + 2b - a\,.$$

VII. The application $f\colon \mathcal{P}(T) \longrightarrow \mathcal{F}_T$ defined through $f(A) = \varphi_A$ is a bijection from the family of the parts belonging to T to the family of the characteristic functions defined on T.

SOLUTION: To demonstrate the injectivity, let $A, B \in \mathcal{P}(T)$ with $A \neq B$. From $A \neq B$ we deduce that there exists $x_0 \in T$ so that:

a) $x_0 \in A$ and $x_0 \notin B$ or b) $x_0 \in B$ and $x_0 \notin A$

In the first case $\varphi_A(x_0) = 1$, case $\varphi_B(x_0) = 0$, so $\varphi_A \neq \varphi_B$. In the second case, also, $\varphi_A \neq \varphi_B$.

The surjectivity comes back to:

$$\forall\, \varphi \in \mathcal{F}_T \quad \exists\, A \in \mathcal{P}(T) \quad \text{a.i.} \quad \varphi = \varphi_A$$

Let then $\varphi \in \mathcal{F}_T$ irrespective. Then for $A = \{x | \varphi(x) = 1\}$ we have $\varphi = \varphi_A$.

II. Functions

Function definition

A function is determined by three elements D, E and f, with the following significations: D and E are sets, named the domain and the codomain, respectively, of the function, and f is a correspondence law from D to E that causes:

each element $\boldsymbol{x} \in \boldsymbol{D}$ to have one,
and only one corresponding element $\boldsymbol{y} \in \boldsymbol{E}$. (F)

So we can say a function is a triplet (D, E, f), the elements of this triplet having the signification stated above.

We usually note this triplet with $f: D \to E$. Two functions are equal if they are equal also as triplets, i.e. when their three constitutive elements are respectively equal (not just the correspondence laws).

To highlight the importance the domain and codomain have in the definition of functions, we will give some examples below, in which, keeping the correspondence law f unchanged and modifying just the domain or (and) the codomain, we can encounter all possible situations, ranging from those where the triplet is a function to those where it is a bijective function.

In order to do this, we have to first write in detail condition (F) that characterizes a function.

We can consider this condition as being made up of two sub conditions, namely:

(f_1) *each element x from the domain has a corresponding element, in the sense of "at least one element", in the codomain.*

Using a diagram as the one drawn below, the proposition can be stated like this:

"(At least) one arrow can be drawn from each point of the domain."

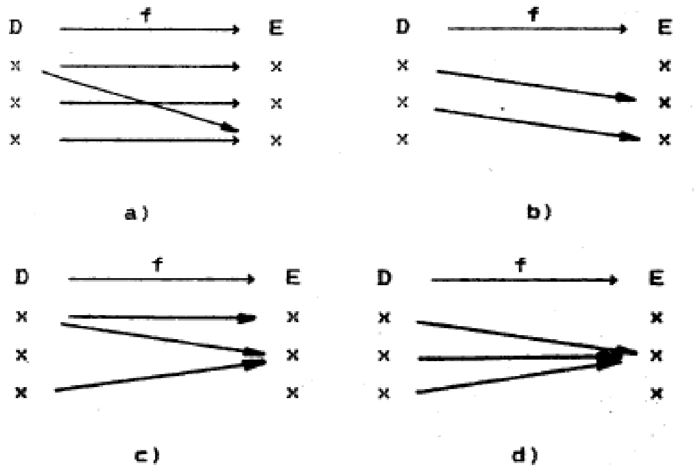


Fig. 2.1

More rigorously, this condition is expressed by:

$$(f_1) \qquad \forall\, x \in D \;\; \exists\, y \in E \,,\; y = f(x)$$

The second sub condition regarding f in the definition of a function is:

(f_2) *the element from the codomain that corresponds to x is unique.*

For the type of diagram in fig. 2.1, this means that **the arrow that is drawn from one point is unique.**

This condition has the equivalent formulation:

"*If two arrows have the same starting point then they also have the same arrival point.*"

Namely,

$$(f_2) \qquad \forall\, x_1 = x_2 \;\; ===> \;\; f(x_1) = f(x_2)$$

Conditions (f_1) and (f_2) are necessary and sufficient for any correspondence law f to be a function. These conditions are easy to use in practice.

In *Figure 2.1,* the correspondence law from a) satisfies (f_1) and doesn't satisfy (f_2); in b) the correspondence law doesn't satisfy (f_1), but satisfies (f_2). In diagram c), neither (f_1), nor (f_2) are satisfied, and in d) the correspondence law satisfies both (f_1) and (f_2), so it is the (only) function.

A function is therefore a triplet (D, E, f), in which the correspondence law f satisfies (f_1) and (f_2).

Let's observe that these two condition only refer to the domain of the function:

(f_1) – from each point of the domain stems at least one arrow;

(f_2) – the arrow that stems from one point of the domain is unique.

It is known that the graphic of a function is made up of the set of pairs of points $(x, f(x))$, when x covers the domain of the function.

$$G_f = \{ (x, f(x)) \mid x \in D \}$$

HOW CAN WE RECOGNIZE ON A GRAPHIC IF A CORRESPONDANCE LAW IS A FUNCTION [if it satisfies (f_1) and (f_2)] ?

In order to answer this question we will firstly remember (for the case $D, E \subset R$) the answer to two other questions:

1. Given x, how do we obtain – using the graphic – $f(x)$ [namely, the *image* of x (or *images*, if more than one, and in this case the correspondence law f is, of course, not a function)] ?

Answer: We trace a parallel from x to $0y$ to the point it touches the graphic, and from the intersection point (points) we then trace a parallel (parallels) to $0x$. The intersection points of these parallels with $0y$ are the images $f(x)$ of x.

2. Reciprocally, given y, to obtain the point (points) x having the property $f(x) = y$, we trace a parallel to $0x$ through y, and

from the point (points) of intersection with the graphic, we then trace a parallel (parallels) to $0y$.

Examples

1. A circle with the center at the origin and with the radius r is not the graphic of a function $f: \mathbb{R} \longrightarrow \mathbb{R}$, because it does not satisfy condition (f_1) (*there are points in the domain that do not have any image*, namely, all the points through which the parallel $0y$ doesn't intersect the graphic) or (f_2), because *there are points in the domain that have more than one image* (all the points $x \in (-r, r)$ have two images).

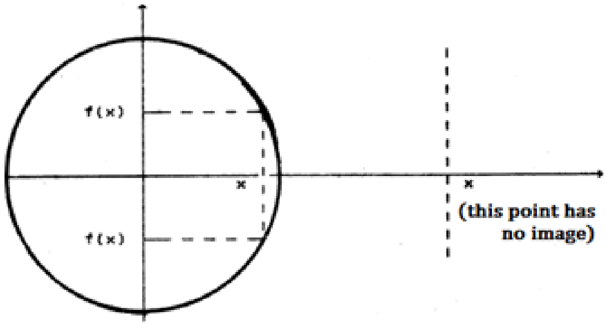


Fig. 2.2

2. A circle with the center at the origin and with the radius r is not the graphic of a function $f: [-r, r] \longrightarrow \mathbb{R}$, because it does not satisfy condition (f_2) (this time there are no more points in the domain that do not have an image, but there are *points that have two images*).

3. A circle with the center at the origin and with the radius r is not the graphic of a function $f: \mathbb{R} \longrightarrow [0, \infty)$, because it does not satisfy condition (f_1). With the codomain $[0, \infty)$, the points have *one image at most*. There are *points, however that don't have an image.*

4. A circle with the center at the origin and with the radius r is the graphic of a function $f:[-r,r] \longrightarrow [0,1]$, because *all the points in the domain have one, and only one image.*

5. A circle with the center at the origin and with the radius r is the graphic of a function $f:[-r,r] \longrightarrow [0,1]$.

In all these examples, the correspondence law has remained unchanged (a circle with the center at the origin and with the radius r, having therefore the equation $x^2 + y^2 = r^2$, which yields $y = \pm(r^2 - x^{2^{1/2}})$.

By modifying just the domain and (or) the codomain, we highlighted all possible situations, starting from the situation where none of the required conditions that define a function were fulfilled, to the situation where both conditions were fulfilled.

WITH THE HELP OF THE PARALLELS TO THE COORDINATES AXES, WE RECOGNIZE THE FULFILLMENT OF THE CONDITIONS (f_1) and (f_2), THUS:

a) A graphic satisfies the condition (f_1) *if and only if any parallel to* $0y$ *traced through the points of the domain touches the graphic in at least one point.*

b) *A graphic satisfies the condition* (f_2) *if and only if any parallel to* $0y$ *traced through the points of the domain touches the graphic in one point at most.*

The inverse of a function

We don't always obtain a function by inverting the correspondence law (inverting the arrow direction) for a randomly given function $f: D \longrightarrow E$. Hence, in the *Figure 2.3*, f is a function, but f^{-1} (obtained by inverting the correspondence law f) is not a function, because it does not satisfy (f_2) (there are points that have more than an image).

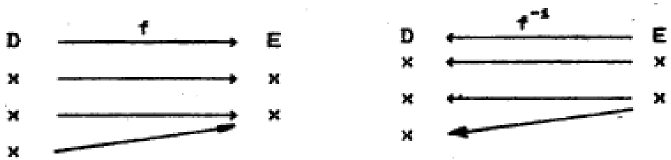


Fig. 2.3

With the help of this diagram, we observe that the inverse doesn't satisfy (f_2) every time there are points in the codomain of f that are the image of at least two points from the domain of f.

In other words, f^{-1} *doesn't satisfy* (f_2) *every time there are different points that have the same image through* f.

Therefore, as by inverting the correspondence law, the condition f still be satisfied, it is necessary and sufficient for the different points through the direct function have different images, namely

$$(f_3) \quad \forall\, x_1, x_2 \in D\,, \quad x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$$

A second situation where f^{-1} is not a function is when it does not satisfy condition (f_1) :

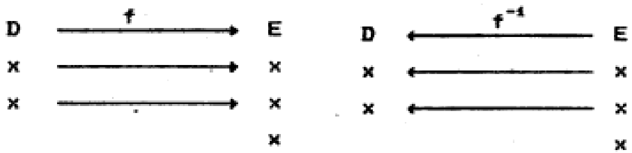


Fig 2.4

We observe this happens every time there are points in the codomain through the direct function that are not the images of any point in the domain.

So as by inverting the correspondence law, the condition (f_1) still be satisfied, it is necessary and sufficient for *all the points in the codomain be consummated through the direct function*, namely

$$(f_4) \quad \forall\, y \in B \quad \exists\, x \in D\,, f(x) = y$$

In conclusion,

f^{-1} satisfies (f_2) if and only if f satisfies (f_3)

f^{-1} satisfies (f_1) if and only if f satisfies (f_4)

f^{-1} is a function if and only if f satisfies $(f_3) + (f_4)$.

As it is known, a function that satisfies (f_3) is called an **injective function**, a function that satisfies (f_4) is called a **surjective function**, and a function that satisfies $(f_3) + (f_4)$ is called a **bijective function**.

We see then that the affirmation: "A function has an inverse if and only if it is bijective", has the meaning that the inverse f^{-1} (that always exists, as a **correspondence law**, even if f is not bijective) **is a function** if and only if f is bijective.

With the help of the parallels to the axes of coordinates, we recognize if a graphic is the graphic of an injective or surjective function; thus:

c) *a graphic is the graphic of an injective function if and only if any parallel to* $0x$ *traced through the points of the codomain touches the graphic in one point, at most [namely* f^{-1} *satisfies* (f_2)*].*

d) a graphic is the graphic of a surjective function if and only if any parallel to $0x$ *traced through the points of the codomain touches the graphic in at least one point [namely* f^{-1} *satisfies* (f_1)*].*

Examples

1. A circle with the center at the origin and with the radius r is the graphic of a function $f: [0, r] \longrightarrow [0, \infty)$ that **is injective** but **not surjective**.

2. A circle with the center at the origin and with the radius r is the graphic of a function $f: [-r, r] \longrightarrow [0, r)$ that **is surjective** but **not injective**.

3. A circle with the center at the origin and with the radius r is the graphic of a **bijective** function $f: [0, r] \longrightarrow [0, r)$.

So, by only modifying the domain and the codomain, using a circle with the center at the origin and with the radius r, all situations can be obtained, starting from the situation where none of the two required conditions that define a function were fulfilled, to a bijective function.

Observation

f^{-1} is obtained by inverting the correspondence law F, namely

$$x \xrightarrow{f} y \quad \Longleftrightarrow \quad x \xleftarrow{f^{-1}} y$$

in other words

$$y = f(x) \quad \Longleftrightarrow \quad x = f^{-1}(y) \qquad (2.1)$$

In the case of the exponential function, for example, the equivalence (2.1) becomes:

$$y = a^x \quad \Longleftrightarrow \quad x = \log_a y \qquad (2.2)$$

because the inverse of the exponential function is noted by $f^{-1}(y) = log_\alpha y$. The relation (2.2) defines the logarithm:

The logarithm of a number y in a given base, a, is the exponent x to which the base has to be raised to obtain y.

The Graphic of the Inverse Function

If D and E are subsets of R and the domain D of f (so the codomain of f^{-1}) is represented on the axis $0x$, and on the axis $0y$, the codomain E (so the domain of f^{-1}), then the graphic of f and f^{-1} coincides, because f^{-1} only inverses the correspondence law (it inverses the direction of the arrows).

But if we represent for f^{-1} the domain, horizontally, and the codomain vertically, so we represent E on $0x$ and D on $0y$, then any random point on the initial graphic G_f (that, without the

aforementioned convention is the graphic for both f and f^{-1}), such a point $(x, f(x))$ becomes $(f(x), x)$.

The points $(x, f(x))$ and $(f(x), x)$ are symmetrical in relation to the first bisector, so we obtain another graphic G_f^a besides G_f **if the domain of f^{-1} is on $0x$.**

Agreeing to represent **the domains of all the functions** on $0x$ it follows that G_f^a is a graphic of f^{-1}.

With this convention, the graphics of f and f^{-1} are symmetrical in relation to the first bisector.

Methods to show that a function is bijective

1. Using the definition

- to study the injectivity, we verify if the function fulfills the condition (f_3);
- to study the surjectivity, we verify if the condition (f_4) is fulfilled.

2. The graphical method

- to study the injectivity we use proposition c);
- to study the surjectivity we use proposition d).

Important observation

If we use the graphical method, it is essential, for functions defined on branches, to trace as accurately as possible the graphic around the point (points) of connection among branches.

Examples

1. The function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \begin{cases} 2x+1 & if\ x \le 1 \\ x+3 & if\ x > 1 \end{cases}$ is injective, but is not surjective. The graphic is represented in *Figure 2.5 a)*.

2. The function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \begin{cases} 3x-1 & if\ x \le 2 \\ 2x-1 & if\ x > 2 \end{cases}$ is surjective, but is not injective. The graphic is in *Figure 2.5 b)*.

3. The function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \begin{cases} 2x+1 & if\ x \le 1 \\ x+2 & if\ x > 1 \end{cases}$ is bijective. The graphic is represented in *Figure 2.5 c)*.

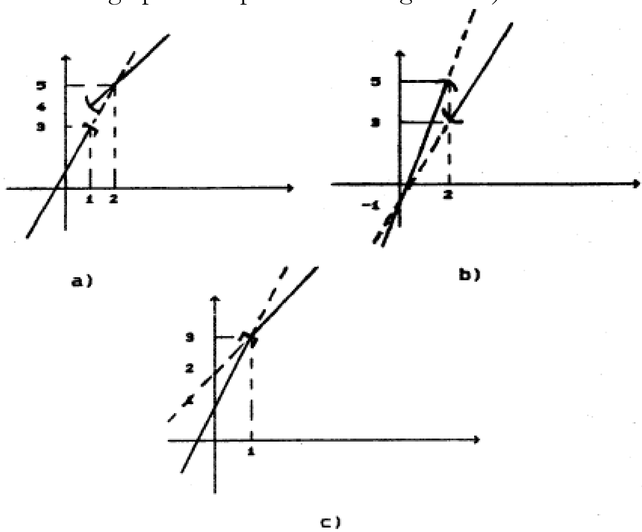


Fig. 2.5

3. Using the theorem

A strictly monotonic function is injective.

To study the surjectivity, we verify if the function is continuous and, in case it is, we calculate the limits at the extremities of the domain.

Example

Let's show that $f(x) = tg\, x,\ f\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow \mathbb{R}$ is bijective.

(i) *injectivity*: because $f(x) = \frac{1}{cos^2 x}$ is strictly positive, we deduce that the function is strictly ascending, so, it is injective.

(ii) *surjectivity*: the function is continuous on all the definition domain, so it has Darboux's property and

$$\lim_{\substack{x \to -\pi/2 \\ x > -\pi/2}} f(x) = -\infty, \quad \lim_{\substack{x \to \pi/2 \\ x < \pi/2}} f(x) = \infty$$

from where it follows that it is surjective.

4. Using the proposition 1

The function , $f\colon D \longrightarrow E$ is bijective if and only if

$\forall\, y \in E$ the equation $f(x) = y$ **has an unique solution** (see the *Algebra workbook grade XII*)

Example

Let $D = \mathbb{Q} \times \mathbb{Q}$ and the matrix $A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$. Then the function

$$f^{*}_{A} : D \longrightarrow D, \quad f^{*}_{A}(x_1, x_2) = (3x_1 - x_2, 4x_1 + 2x_2)$$

is bijective *(Algebra, grade XII).*

Let's observe that in the proposition used at this point, the affirmation "the equation $f(x) = y$ has a solution" ensures the surjectivity of the function and the affirmation "the solution is unique" ensures the injectivity.

5. Using the proposition 2

If $f, g\colon D \longrightarrow D$ and $g.f = 1_D$ then f is injective and g is surjective *(Algebra, grade XII).*

Example

Let $D = \mathbb{Z} \times \mathbb{Z}$ and $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$. Then the function $f_A\colon D \longrightarrow D$ defined through

$$f_A(x_1, x_2) = (2x_1 + 3x_2 \, , \, -x_1 - 2x_2)$$

satisfies $f_A \circ f_A = 1_D$, so it is bijective (algebra, grade XII).

Let’s observe that the used proposition can be generalized, at this point, thus:

“Let $f: D \longrightarrow E, g: E \longrightarrow F$, so that $g \circ f$ is bijective. Then f is injective and g is surjective.”

Exercises

I. Trace the graphics of the following functions and specify, in each case, if the respective function is injective, surjective or bijective.

1. $f(x) = \min\ (x + 1 \, , \, x^2 + 2 \, , \, 3x)$

2. $f(x) = \max\ (3x - 1 \, , \, 2x + 3 \, , \, x^2 - 2x)$

3. $f(x) = \min\ (2x - 3, \, 4x^2 - 5 \, , \, [x])$

4. $f(x) = \min\ (|x - 3| \, , \, 4x)$

5. $f(x) = \min\limits_{-1 \le t \le x} t^2$

6. $f(x) = \inf\limits_{t < x} (t^2 + 2t + 3)$

7. $f(x) = \inf\limits_{t < x} \dfrac{(t + 1)^2}{t^2 + 1}$

8. $f(x) = \min\limits_{-2 \le t \le x} (t^2 - |t - 2|)$

9. $f(x) = \min\limits_{-2 < t < x} \dfrac{t^4}{(t + 7)^3}$

10. $f(x) = \sup\limits_{0 < t < x} t^2 \ln t$

11. $f(x) = \max\limits_{-3 \le t \le x} (t - 2 \operatorname{arctg} t)$

12. $f(x) = \max\limits_{-\pi/4 \le t \le x} \sin t$

SOLUTIONS.

It is known that:

$$\min(u(x),v(x),w(x)) =$$

$$= \begin{cases} u(x) \text{ if } u(x) \leq v(x) \text{ and } u(x) \leq w(x) \\ v(x) \text{ if } v(x) \leq u(x) \text{ and } v(x) \leq w(x) \\ w(x) \text{ if } w(x) \leq u(x) \text{ and } w(x) \leq v(x) \end{cases}$$

In order to more easily explain the conditions from the inequalities over u, v *and* w we will proceed as follows:

1. we draw the table with the signs of the functions $u - v, u - w$ and $v - w$

2. using the table we can easily explain the inequalities, because for example $u(x) \leq v(x) <===> u(x) \leq 0$.

1. For $u(x) = x + 1$, $v(x) = x^2 + 2$ and $w(x) = 3x$ we have the following table:

x		$\frac{1}{2}$		1		2	
$u(x)-v(x)$	$-$		$-$	$-$	$-$	$-$	$-$
$u(x)-w(x)$	$+$	0	$-$	$-$	$-$	$-$	$-$
$v(x)-w(x)$	$+$		$+$	0	$-$ $-$	0	$+$ $+$

So:

a) $x \in \left(-\infty , \frac{1}{2}\right] \longrightarrow u(x) \leq v(x)$

and $u(x) \geq w(x)$, so $f(x) = w(x)$

b) $x \in \left(\frac{1}{2} , 1\right] \longrightarrow u(x) \leq v(x)$

and $u(x) \leq w(x)$, so $f(x) = u(x)$

c) $x \in (1 , 2] \longrightarrow u(x) \leq v(x)$

and $u(x) \leq w(x)$, so $f(x) = u(x)$

d) $x \in (2, \infty) \longrightarrow u(x) \le v(x)$ and $u(x) \le w(x)$, so $f(x) = u(x)$

Therefore,

$$f(x) = \begin{cases} 3x & \text{if } x \le 1/2 \\ x + 1 & \text{if } x > 1/2 \end{cases}$$

5. To explain f we proceed as follows:

(1) we draw the variation table of the function $y(t) = t^2$

(2) considering x in the first monotonic interval (deduced from the table), at the right of -1 (because we are only interested in the values $t \ge -1$), we calculate the minimum of $y(t)$ on the interval $t \in [-1, x]$ etc.

t	$-\infty$ $\quad$ -1	x	0	x	$+\infty$
y'		$-$	$-$ 0 $+$ $+$	$+$ $+$ $+$	
y		1 ↘	↘ 0 ↗	↗ ↗	

a) for the first monotonic interval, $x \in (-1,0)$, the function $y(t) = t^2$ has a single minimum point on the interval $[-1, x]$, situated in $t = x$; its values is $y(x) = x^2$. Being a single minimum point it is also the global minimum (absolute) of $y(t)$ on the interval $[-1, x]$, so, the value of f is $f(x) = x^2$ for $x \in (-1,0]$.

b) for the second monotonic interval, $x \in (0, \infty)$, the function $y(t) = t^2$ has a single minimum point on the interval $[-1, x]$, in $t = 0$; its value is $y(0) = 0$. Being a single minimum point, we have $f(x) = 0$ for $x \in (0, \infty)$, so

$$f(x) = \begin{cases} x^2 & \text{if } x \in (-1, 0] \\ 0 & \text{if } x \in (0, \infty) \end{cases}$$

Let's observe that using the same variation table we can explain the function $g(x) = \max_{1 \le 1 \le x} t^2$. Hence:

a) for $x \in (-1,0]$, the function $y(t) = t^2$, has only one maximum, namely $[-1, x]$ in $t = -1$; its value is $M = y(1) = 1$. So $g(x) = 1$, for $x \in (1,0]$.

b) for $x \in (0, +\infty)$, $y(t) = t^2$ has two maximum points, in $t_1 = -1$ and $t_2 = x$; their values are $M_1 = y(1) = 1$ and $M_2 = y(x) = x^2$. The values of f is the global maximum, so:

$g(x) = \max(1, x)$ for $x \in (0, +\infty)$.

It follows that $g(x) = \begin{cases} 1 & if\ x \in [-1,0] \\ \max(1, x^2) & if\ x > 0 \end{cases}$

$$\text{So } g(x) = \begin{cases} 1 & \text{if} \quad x \in (-1, 0] \\ 1 & \text{if} \quad x \in (0, 1] \\ x^2 & \text{if} \quad x > 1 \end{cases} =$$

$$= \begin{cases} 1 & \text{if} \quad x \in (-1, 1) \\ x^2 & \text{if} \quad x > 1 \end{cases}$$

9. From the variation table of $y(t) = \frac{t^4}{(t+7)^3}$

t	-2		x		0		x		$+\infty$
$y'(t)$		$-$ $-$		$-$ $-$	0	$+$ $+$		$+$ $+$	
$y(t)$	$\frac{16}{125}$	$\searrow$		$\searrow$	0	$\nearrow$		$\nearrow$ $\nearrow$	

1. for $x \in (-2, o]$, $y(t)$ has a single minimum on the interval $[-2, x]$, in $t = x$; its value is $m = y(x) = \frac{x^4}{(x+7)^3}$

2. for $x > 0$, $y(t)$ has a single minimum, namely $[-2, x]$, in $t = 0$; its value is $m = y(0) = 0$. So

$$f(x) = \begin{cases} \dfrac{x^4}{(x+7)^3} & \text{if} \quad x \in (-2, 0] \\ 0 & \text{if} \quad x > 0 \end{cases}$$

Analogously, for

$$g(x) = \max_{-2 \le t \le x} \frac{t^4}{(t+7)^3} \text{ we have}$$

$$g(t) = \begin{cases} \dfrac{16}{125} & \text{if } x \in (-2, 0] \\ \max\left(\dfrac{16}{125}, \dfrac{t^4}{(x+7)^3}\right) & \text{if } x > 0 \end{cases}$$

10. From the variation table of $y(t) = t^2$ in t

t	0		x			$1/\sqrt{e}$		x		$+\infty$
$y'(t)$	$-$	$-$		$-$	$-$	0	$+$	$+$	$+$	$+$
$y(t)$	0	$\searrow$		$\searrow$		$-1/2e$	$\nearrow$		$\nearrow$	

we deduce:

(a) if $x \in (0, \frac{1}{\sqrt{e}}]$, the function $y(t) = t^2$ in t has, for $t \in (0, x]$, a single supremum $S = \lim_{\substack{t \to o \\ t > 0}} y(t) = 0$.

(b) $x \in (\frac{-1}{\sqrt{e}}, \infty]$, the function $y(t)$ has two supreme values: $S_1 = 0$ and $M_1 = y(x) = x^2 1n\, x$.

So:

$$f(x) = \begin{cases} 0 & x \in \left(0, \dfrac{-1}{\sqrt{e}}\right] \\ \max(0, x^2 \ln x) & x \in \left(\dfrac{-1}{\sqrt{e}}, \infty\right) \end{cases}$$

II. Study the bijectivity of:

$$1.\ f(x) = \begin{cases} 0 & x = 0 \\ \dfrac{1}{x} & x \in Q,\ x \neq 0 \\ x & x \in R - Q \end{cases}$$

2. $f(x) = P(x)$, P being an uneven degree polynomial.

3. $f(x) = \log_a(x + \sqrt{x^2+1})$ $a > 0,\ a \neq 1$.

4. $f(x) = \begin{cases} 2x+1 & x \in \mathbb{Q} \\ \dfrac{2}{x} & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

5. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $f(x) = A \cdot x$

where $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$

6. $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ $f(x) = A \cdot x$

where $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{pmatrix}$

7. $f : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+$

$f(x,y) = (\lg x + 2 \cdot \lg y,\ 3 \cdot \lg x - 2 \cdot \lg y)$

III. Study the irreversibility of the hyperbolic functions:

1. $sh(x) = \dfrac{e^x - e^{-x}}{2}$ **(hyperbolic sine)**

2. $ch(x) = \dfrac{e^x + e^{-x}}{2}$ **(hyperbolic cosine)**

3. $th(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ **(hyperbolic tangent)**

4. $cth(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ **(hyperbolic cotangent)**

and show their inverses.

IV. 1. Show that the functions $f(x) = x^2 - x + 1$, $f: \left[\frac{1}{2}, \infty\right) \longrightarrow \mathbb{R}$ and

$g(x) = \dfrac{1}{2} + \sqrt{x - \dfrac{3}{4}}$, $g : \left[\dfrac{1}{2}, \infty\right) \longrightarrow \mathbb{R}$

are inverse one to the other.

2. Show that $f(x) = \frac{1-x}{1+x}$ coincides with its inverse.

3. Determine the parameters a, b, c, d so that $f(x) = \frac{ax+b}{cx+d}$ coincides with its inverse.

4. Show that the function $f: (0,1] \longrightarrow \mathbb{R}$ defined through$f(x) = \frac{4}{3^n} - x$, if $x \in \left[\frac{1}{3^n}, \frac{1}{3^{n-1}}\right]$ is bijective.

5. On what subinterval is the function $f: [-\frac{1}{2}, \infty) \longrightarrow \mathbb{R}$ $f(x) = \sqrt{x - \sqrt{2x-1}}$ bijective?

INDICATIONS.

4. To simplify the reasoning, sketch the graphic of function f.

5. Use the superposed radical decomposition formula:

$$\sqrt{A \pm \sqrt{B}} = \sqrt{\frac{A+C}{2}} \pm \sqrt{\frac{A-C}{2}},$$

where $C = \sqrt{A^2 - B}$.

Monotony and boundaries for sequences and functions

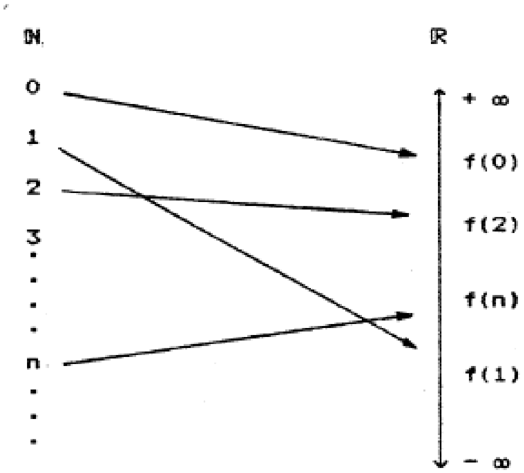
A sequence is a function defined on the natural numbers' set. The domain of such a function is, therefore, the set of natural numbers $\mathbb{N}$. The codomain and the correspondence law can randomly vary.

Hereinafter, we will consider just natural numbers' sequences, i.e. sequences with the codomain represented by the set of real numbers, $\mathbb{R}$.

$f: \mathbb{N} \longrightarrow \mathbb{R}$ REAL NUMBERS SEQUENCE

For this kind of function, with the same domain and codomain each time, only the correspondence law f should be added, which comes down to specifying the set of its values:

$$f(0), f(1), f(2), \dots, f(n), \dots$$



For the ease of notation, we write, for example:

$$f(0) = a_1 \ , \ f(1) = a_2 \ , \ \dots \ , \ f(n) = a_n \ , \ \dots$$

So, to determine a sequence, it is necessary and sufficient to know the set of its values: $a_0, a_1, \dots, a_n, \dots$. This set is abbreviated by $(a_n)_{n \in \mathbb{N}}$.

Consequently, a sequence is a particular case of function. The transition from a function to a sequence is made in this manner:

(s_1) by replacing the domain $D$with $\mathbb{N}$

(s_2) by replacing the codomain E with $\mathbb{R}$

(s_3) by replacing the variable x with n (or m, or i, etc)

(s_4) by replacing $f(x)$ with a_n

	DOMAIN	CODOMAIN	VARIABLE	CORRESP. LAW
Function	D	E	x	$f(x)$
Sequence	$\mathbb{N}$	$\mathbb{R}$	n	a_n

Hereinafter we will use this transition method, from a function to a sequence, in order to obtain the notations for **monotony**, **bounding** and **limit of a sequence** as particular cases of the same notations for functions.

We can frequently attach a function to a sequence $(a_n)_{n\in\mathbb{N}}$*, obtained by replacing* n *with* x *in the expression of* a_n *(*a_n *becomes* $f(x)$*).*

Example

We can attach the function $f(x)=\frac{x+3}{2x+1}$ to the sequence $a_n=\frac{n+3}{2n+1}$.

However, there are sequences that we cannot attach a function to by using this method. For example:

$$a_n = \frac{n!}{n+1}.$$

It is easier to solve problems of monotony, bounding and convergence of sequences by using the function attached to a sequence, as we shall discover shortly.

Using functions to study the monotony, the bounding and the limits of a sequence offers the advantage of using the derivative and the table of variation.

On the other hand, it is useful to observe that the monotony, as well as the bounding and the limit of a sequence are particular cases of the same notion defined for functions. This particularization is obtained in the four mentioned steps $(s_1)-(s_4)$.

Here is how we can obtain the particularization, first for the monotony and bounding, then for the limit.

MONOTONIC FUNCTIONS	MONOTONIC SEQUENCES
a) the function $f: D \longrightarrow \mathbb{R}$, $D \subseteq \mathbb{R}$ is monotonically increasing if: $\forall\, x_1 . x_2 \in D, x_1 \leq x_2 \rightarrow f(x_1) \leq f(x_2)$	a) the sequence $f: \mathbb{N} \longrightarrow \mathbb{R}$ is monotonically increasing if: $\forall\, n_1, n_2 \in \mathbb{N}, n_1 \leq n_2 \rightarrow a_{n_1} \leq a_{n_2}$ (1) If condition (1) is fulfilled, taking, particularly $n_1 = n$ and $n_2 = n + 1$, we deduce $a_n \leq a_{n+1}$ for any n (2) In conclusion $(1) \longrightarrow (2)$ The reciprocal implication also stands true. Its demonstration can be deduced from the following example: If (2) is fulfilled and we take $n_1 = 7, n_2 = 11$, we have $n_1 \leq n_2$ and step by step $a_7 \leq a_8, a_8 \leq a_9, a_9 \leq a_{10}$ and $a_7 \leq a_{11}$. Consequently, $(1) \leftrightarrow (2)$. Because conditions (1) and (2) are equivalent, and (2) is more convenient, we will use this condition to define the monotonically increasing sequence. But we must loose from sight the fact that it is equivalent to that particular condition that is obtained from the definition of monotonic functions, through the particularizations $(s_1) - (s_4)$, that define the transition from function to sequence.

b) the function $f: D \longrightarrow \mathbb{R}$, $D \subseteq \mathbb{R}$ is monotonically decreasing if: $\forall\, x_1, x_2 \in D, x_1 \leq x_2 \rightarrow f(x_1) \geq f(x_2)$	b) the sequence $f: \mathbb{N} \longrightarrow \mathbb{R}$ is monotonically decreasing if: $\forall\, n_1, n_2 \in \mathbb{N}, n_1 \leq n_2 \rightarrow a_{n_1} \geq a_{n_2}$ It can be shown that this condition is equivalent to: $a_n \geq a_{n+1}$ for any $n \in \mathbb{N}$ (4)
LIMITED FUNCTIONS	LIMITED SEQUENCES
a) $f: D \longrightarrow \mathbb{R}$, $D \subseteq \mathbb{R}$ is of **inferiorly bounded** if it doesn't have values towards $-\infty$, i.e. $\exists a \in \mathbb{R}, \forall\, x \in D, f(x) \geq a$ b) $f: D \longrightarrow \mathbb{R}$, $D \subseteq \mathbb{R}$ is of **superiorly bounded** if it doesn't have values towards $+\infty$ i.e. $\exists b \in \mathbb{R}, \forall\, x \in D, f(x) \leq b$ c) $f: D \longrightarrow \mathbb{R}$, $D \subseteq \mathbb{R}$ is **bounded**, if it is inferiorly and superiorly bounded, i.e. $\exists a, b \in \mathbb{R}, \forall\, x \in \mathbb{N}\ a \leq f(x) \leq b$. In other words, there exists an interval $[a, b]$ that contains all the values of the function. This interval is not, generally, symmetrical relative to the origin, but we can consider it so, by enlarging one of the extremities wide enough. In this case $[a, b]$ becomes $[-M, M]$ and the limiting condition is: $\exists\, M > 0, , \forall\, x \in D \quad -M \leq f(x) \leq M$ i.e. $\exists\, M > 0, , \forall\, x \in D\ \ \lvert f(x) \rvert \leq M$	a) the sequence $f: \mathbb{N} \longrightarrow \mathbb{R}$ is of **inferiorly bounded** if it doesn't have values towards $-\infty$, i.e. $\exists a \in \mathbb{R}, \forall\, n \in \mathbb{N}, a_n \geq a$ b) the sequence $f: \mathbb{N} \longrightarrow \mathbb{R}$ is of **superiorly bounded** if it doesn't have values towards $+\infty$, i.e. $\exists b \in \mathbb{R}, \forall\, n \in \mathbb{N}, a_n \geq b$ c) the sequence $f: \mathbb{N} \longrightarrow \mathbb{R}$ is **bounded** if it is inferiorly and superiorly bounded, i.e $\exists a, b \in \mathbb{R}, \forall\, n \in \mathbb{N}\ \ a \leq a_n \leq b$ OR $\exists\, M > 0, \forall\, n \in \mathbb{N}\ \ \lvert a_n \rvert \leq M$

Methods for the study of monotony and bounding

METHODS FOR FUNCTIONS	METHODS FOR SEQUENCES
The study of monotony	
1. **Using the definition**: We consider $x_1 \leq x_2$ and we compare the difference $f(x_1) = f(x_2)$ to zero. This can be done through successive **minorings** and **majorings** or by applying **Lagrange**'s theorem to function f on the interval $[x_1, x_2]$. 2. **Using the variation table** (in the case of differentiable functions) As it is known, **the variation table** of a differentiable function offers precise information on **monotonic and bounding** functions. 3. **Using Lagrange's theorem** It allows the replacement of the difference $f(x_2) - f(x_1)$ with $f(c)$ that is then compared to zero.	1. Using the definition: We compare the difference $a_{n+1} - a_n$ to zero, and for sequences with positive terms we can compare the quotient a_{n+1}/a_n to one. We can make successive **minorings** and **majorings** or by applying **Lagrange**'s theorem to the attached considered sequence. 2. **Using the variation table for the attached function** We study the monotony of the given sequence and, using the **sequences criterion**, we deduce that the monotony of the sequence is given by the monotony of this function, on the interval $[0, \infty)$, (**see Method 10 point c**) 3. **Using Lagrange's theorem for the attached function.**
The study of bounding	
1. **Using the definition** 2. **Using the variation table**	1.. **Using the definition** 2. **Using the variation table for the attached function** 3. **If the sequence**

	decomposes in a finite number of bounded subsequences, it is bounded. 4. **Using the monotony**. If a sequence is monotonic, at least half of the bounding problem is solved, namely: a) **if the sequence is monotonically increasing**, it is inferiorly bounded by the first term **and only the superior bound has to be found**. **b) if the sequence is monotonically decreasing,** it is superiorly bounded by the first term, and **only the inferior bound must be found**. c) IF THE SEQUENCE IS MONOTONICALLY DECREASING AND HAS POSITIVE TERMS, **IT IS BOUNDED.**

Exercises

I. Study the monotony and the bounding of the functions:

1. $f(x)=\begin{cases} x+1 & \text{if } x \le 1 \\ x^2+x+1 & \text{if } x > 1 \end{cases}$

2. $f(x) = x^4 + |3x^2 - 4|$

3. $f(x) = (\ln x)^{\ln x}$, $f : [1,\infty) \longrightarrow \mathbb{R}$

4. $f(x) = \dfrac{mx+n}{\sqrt{x^2+x+1}}$, $f : \mathbb{R} \longrightarrow \mathbb{R}$

5. $f(x) = \sqrt{x - 2 - \sqrt{2x-5}}$,

$f : [3,\infty) \longrightarrow \mathbb{R}$

6. $f(x) = \text{arctg}\, x$, $f : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

ANSWERS:

3. We can write $f(x) = e^{ln \times \ln(\ln x)}$ and we have the variation table:

x	1				$e^{1/e}$					
$f'(x)$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$
$f(x)$	1	$\searrow$			$e^{-1/e}$		$\nearrow$			

so the function is decreasing on the interval $(1, e^{1/e})$ and increasing on $(e^{1/e}, \infty)$. It is inferiorly bounded, its minimum being $m = e^{-1/e}$ and it is not superiorly bounded.

5. We use the superposed radicals' decomposition formula:

$$f(x) = \sqrt{A + \sqrt{B}} = \sqrt{\frac{A + C}{2}} - \sqrt{\frac{A - C}{2}},$$

with $C = A^2 - B$

6. Let $x_1 < x_2$. In order to obtain the sign of the difference $f(x_1) - f(x_2)$ we can apply Lagrange's theorem to the given function, on the interval $[x_1, x_2]$ (the conditions of the theorem are fulfilled): $c \in \mathbb{R}$ exists, so that:

$$f(x_1) - f(x_2) = f'(c)(x_1 - x_2) =$$

$$= \frac{1}{1 - c^2}(x_1 - x_2) < 0$$

so the function is increasing.

II. Study the monotony and the bounding of the sequences:

1. $a_n = \frac{\ln n}{n}$

2. $a_n = \sqrt[n]{n}$

also determine the smallest term of the sequence.

$$3.\quad a_n = n\,a^n$$

also determine the smallest term of the sequence.

$$4.\quad a_n = \frac{n-1}{\ln n} - \sqrt{n}\ ,\ \text{for } n \geq 2\ .$$

$$5.\quad a_n = \left(1+\frac{1}{n}\right)^n,\ b_n = \left(1+\frac{1}{n}\right)^{n+1},$$

$$c_n = \left(1+\frac{1}{n}\right)^{n+\alpha},\ \text{with}$$

$$\alpha \in (0\ ,\ 1)\ ,\ d_n = \left(1+\frac{1}{n}\right)^{\gamma_n}\ ,$$

γ_n being the intermediary value that is obtained by applying Lagrange's theorem to the function $f(x) = \ln x$ on the intervals $[n, n+1]$.

ANSWERS:

2. The function attached to the sequence is $f(x) = x^{1/x}$. From the variation table:

x	2			e			3	
$f'(x)$		+	+	0	−	−	−	−
$f(x)$		↗	↗	$f(e)$	↘		↘	

we deduce, according to the sequence criterion (Heine's criterion, see *Method 10, point c*), the type of monotony of the studied sequence. The sequence has the same monotony as the attached function, so it is decreasing for $n \geq 3$. Because the sequence is decreasing if $n \geq 3$, in order to discover the biggest term, we have to compare a_2 and a_3. We deduce that the biggest term of the sequence is:

$$a_3 = \sqrt[3]{3}\ .$$

3. The function attached to the sequence is $f(x) = xa^x$. From the variation table for $a \in (0,1)$:

x	0			$(-\ln a)^{-1}$				
$f'(x)$		$+$	$+$	0	$-$	$-$	$-$	$-$
$f(x)$		$\nearrow$	$\nearrow$		$\searrow$		$\searrow$	

using point c) from Method 10, we deduce that the given sequence is increasing for n, smaller or equal to the integer part of $(-\ln a)^{-1}$ and is decreasing for $n \geq [(-\ln a)^{-1}] + 1$.

Because

$$\lim_{\substack{a \to 0 \\ a > 0}} (-\ln a)^{-1} = 0 \text{ and } \lim_{\substack{a \to 1 \\ a < 1}} (-\ln a)^{-1} = +\infty ,$$

we deduce that the interval $\left[1, [(-\ln a)^{-1}]\right]$, in which the sequence is increasing, **can be no matter how big or small**.

6. The sequence of the general term a_n is increasing, the sequence of the general term b_n is decreasing, and the sequence c_n is increasing if $a \in [0, n - \gamma_n]$ and is decreasing for $a \in [, n - \gamma_n, 1]$.

III. Using Lagrange's theorem.

1. Prove the inequality $|\sin x| \leq |x|$ for any real x, then show that the sequence $a_1 = \sin x, a_2 = \sin\sin x, \dots, a_n = \sin\sin \dots \sin x$, is monotonic and bounded, irrespective of x, and its limit is zero.

2. Study the monotony and the bounding of the sequences:

$$a_n = \left(1 + \frac{1}{n}\right)^n, \; b_n = \arcsin\left(\frac{2\sqrt{n}}{1+n}\right),$$

$$c_n = \frac{\ln n}{n^2 + n}, \; d_n = \frac{n+1}{2n+1} + 2\ln n,$$

$$e_n = n \cdot \sqrt{\frac{n+2}{n+3}}$$

ANSWERS:

1. $|\sin x| \leq |x| \iff \left|\frac{\sin x}{x}\right| \leq 1 .$

From Lagrange's theorem applied to the function $f(n) = sin\, t$ on the interval $[x, 0]$ or $[0, x]$, depending on x being either negative or positive, we obtain $\left|\frac{\sin x}{x}\right| = |\cos c| \leq 1$. The sequence satisfies the recurrence relation: $a_{n+1} = \sin a_n$, so if $\sin x > 0$, we deduce that the sequence is decreasing, and if $\sin x < 0$ the sequence is increasing. It is evidently bounded by -1 and 1. Noting with L its limit and passing to the limit in the recurrence relation, we obtain $L = \sin L$, so $L = 0$.

$$2.\ \text{a)}\ a_{n+1} - a_n = \left(1 + \frac{1}{n+1}\right)^{n+1} - \left(1 + \frac{1}{n}\right)^n = \frac{f(n+1) - f(n)}{n+1-n},$$

where $f(x) = (1 + \frac{1}{x})^x$ defines the function attached to the sequence.

b) Applying Lagrange's theorem to the function attached to the sequence, on the interval $[n, n+1]$, we deduce $a_{n+1} - a_n = f(c_n)$. We have to further determine the sign of the derivation:

$$f'(x) = \left(1 + \frac{1}{x}\right)^x \cdot \left[\ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1}\right]$$

namely, the sign of

$$g(x) = \ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1}.$$

From the table below we deduce $g(x) > 0$ for $x > 0$, so $a_{n+1} - a_n > 0$.

x	0					$+\infty$
$g'(x)$	$-$	$-$	$-$	$-$	$-$	
$g(x)$	$\searrow$		$\searrow$		$\searrow$	

IV. 1. Let I be a random interval and $f: I \to I$ a function. Show that the sequence defined through the recurrence relation $a_{n+1} = f(a_n)$, with a_o being given, is:

a) increasing, if $f(x) - x > 0$ on I

b) decreasing if $f(x) - x < 0$ on I.

2. If the sequence $(a_n)_{n\in\mathbb{N}}$ is increasing, and the sequence $(b_n)_{n\in\mathbb{N}}$ is decreasing and $a_n \le b_n$ for any n, then:

a) the two sequences are bounded, and so convergent.

b) if $\lim\limits_{n\to\infty}(b_n - a_n) = 0$ then they have the same limit.

c) apply these results to the sequences given by the recurrence formulas:

$$a_{n+1} = \sqrt{a_n \cdot b_n}\ ,\ b_{n+1} = \frac{a_n + b_n}{2}$$

with a_o and b_o being given.

SOLUTIONS:

1. $a_{n+1} - a_n = f(a_n) - a_n > 0$ in the first case.

2. the two sequences are bounded between a_1 and b_1, so the sequences are convergent, because they are also monotonic. Let l_1 and l_2 represent their limits. From the hypothesis from (b) it follows that $l_1 = l_2$.

(c) the sequences have positive terms and:

$$b_n^2 - a_n^2 = \left(\frac{b_{n-1} - a_{n-1}}{2}\right)^2 > 0\ .$$

V. Study the bounding of the sequences:

1. $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{5}, \ldots,$
$, \frac{1}{2^n}, \frac{1}{n+2}, \ldots$

2. $\sin 1, \frac{1}{2}, \sin\left(\frac{1}{2}\right), \frac{1}{2}, \ldots,$
$, \sin\left[\frac{1}{n^2}\right], \frac{1}{2^n}, \ldots$

3. $1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \ldots$

4. $2, 4, \frac{9}{4}, \frac{27}{8}, \ldots,$
$, \left(1 + \frac{1}{n}\right)^n, \left(1 + \frac{1}{n}\right)^{n+1}$

5. $\cos(\pi/4)$, $\cos(\cos(\pi/4))$, ...

6. 1 , 2 , $\sqrt{2}$, $\frac{3}{2}$, $\sqrt{3}$, ... , a_n, b_n, ...

with $a_0 = 1$, $b_0 = 2$, $a_{n+1} = \sqrt{1 + a_n}$,

, $b_{n+1} = \dfrac{1 + b_n}{2}$

ANSWER: Each sequence is decomposed in two bounded subsequences (convergent even, having the same limit), so they are bounded (convergent).

III. The limits of sequences and functions

The definition of monotony and bounding of sequences has been deduced from the corresponding definitions of functions, using the method shown, in which x is replaced with n and $f(x)$ is replaced with a_n. We will use the same method to obtain the definition of the limit of a sequence from the definition of the limit of a function. This method of particularization of a definition of functions with the purpose of obtaining the analogous definition for sequences highlights the connection between sequences and functions. Considering that sequences are particular cases of functions, it is natural that the definitions of monotony, bounding and the limit of a sequence are particular cases of the corresponding definitions of functions.

The limits of functions

The definition of a function's limit is based on the notion of vicinity of a point.

Intuitively, the set $V \subset \mathbb{R}$ is in the vicinity of point $x_o \in \mathbb{R}$, if (1) $x \in V$, and, moreover, (2) V also contains points that are in the vicinity of x_o (at its left and right).

Obviously, if there is any open interval (a, b), so that $x_o \in (a, b) \subset V$, then the two conditions are met.

Particularly, any open interval (a, b) that contains x_o satisfies both conditions, so it represents a vicinity for x_o.

For finite points, we will consider as vicinities open intervals with the center in those points, of the type:

$$V_l = (l - \varepsilon, l + \varepsilon) \quad V_{x_o} = (x_o - \delta, x_o + \delta)$$

and sets that contain such intervals.

For $(+\infty)$, $V_\infty = (\infty - \varepsilon, \infty + \varepsilon)$ doesn't make sense, so we have to find a different form for the vicinities of $+\infty$. In order to accomplish this, we observe that to the right of $+\infty$ we cannot consider any points, so $\infty + \varepsilon$ has to be replaced with $+\infty$. Furthermore, because $\infty - \varepsilon = \infty$, we replace $\infty - \varepsilon$ with ε. So, we will consider the vicinities of $+\infty$ with the form:

$$V_\infty = (\varepsilon, \infty)$$

and sets that contain such intervals.

Analogously, the vicinities of $-\infty$ have the form:

$$V_{-\infty} = (-\infty, \varepsilon)$$

The definition of a function's limit in a point expresses the condition according to which when x approaches x_o, $f(x)$ will approach the value l of the limit.

With the help of vicinities, this condition is described by:

$$\lim_{x \to x_o} f(x) = l \Longleftrightarrow$$

$$\Longleftrightarrow [\forall V_l \; \exists V_{x_o} \; \forall x \in V_{x_o}, \; x \neq x_o \Rightarrow f(x) \in V_l] \; (***)$$

As we have shown, for the vicinities V_l and V_{x_o} from the definition $(***)$, there exist three essential forms, corresponding to the finite points and to $+\infty$ and $-\infty$ respectively.

CONVENTION:

We use the letter ε to write the vicinities of l, and the letter δ designates the vicinities of x_o.

We thus have the following cases:

a) x_o – finite $\Rightarrow V_{x_o} = (x_o - \delta, x_o + \delta)$

$$\text{and } x \in V_{x_o} \Leftrightarrow x \in (x_o - \delta,\ x_o + \delta) \Leftrightarrow$$

$$\Leftrightarrow x_o - \delta < x < x_o + \delta \Leftrightarrow$$

$$\Leftrightarrow -\delta < x - x_o < \delta \Leftrightarrow |x - x_o| < \delta$$

$$\text{b)} \quad x_o = +\infty \Rightarrow V_{x_o} = V_{\infty} = (\delta,\ +\infty)$$

and condition $x \in$ and V_{x_o} becomes $x > \delta$

$$\text{c)} \quad x_o = -\infty \Rightarrow V_{x_o} = V_{-\infty} = (-\infty,\ \delta)$$

and condition $x_o \in$ and V_{x_o} becomes $x < \delta$

$$\text{a')} \ l - \text{finit} \Rightarrow V_l = (l - \varepsilon,\ l + \varepsilon)$$

so the condition $f(x) \in V_l$ becomes:

$$f(x) \in (l - \varepsilon, l + \varepsilon) \Leftrightarrow$$

$$\Leftrightarrow l - \varepsilon < f(x < l + \varepsilon \Leftrightarrow$$

$$\Leftrightarrow -\varepsilon < f(x) - l < \varepsilon \Leftrightarrow |f(x) - l| < \varepsilon$$

$$\text{b')} \ l = +\infty \Rightarrow V_l = V_{+\infty} = (\varepsilon,\ +\infty)$$

and the condition $f(x) \in V_l$ becomes $f(x) > \varepsilon$

$$\text{c')} \ l = -\infty \Rightarrow V_l = V_{-\infty} = (-\infty,\ \varepsilon)$$

and the condition $f(x) \in V$ becomes $f(x) < \varepsilon$

All these cases are illustrated in *Table 3.1*.

Considering each corresponding situation to x_o and l, in the definition $(***)$, we have 9 total forms for the definition of a function. we have the following 9 situations:

(l_1) x_o – finite , l – finite

$$\lim_{x \to x_o} f(x) = l \Leftrightarrow [\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0\ \forall x \neq x_o,$$

$$|x - x_o| < \delta \Rightarrow |f(x) - 1| < \varepsilon]$$

(l_2) $x_o = +\infty$, l – finite

$$\lim_{x \to \infty} f(x) = l \iff$$

$$\iff [\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0\ \forall x > \delta_\varepsilon \Rightarrow |f(x) - l| < \varepsilon]$$

(l_3) $x_0 = -\infty$, l – finite

$$\lim_{x \to -\infty} f(x) = l \iff$$

$$\iff [\forall \varepsilon > 0\ \exists \delta_\varepsilon\ \forall x < \delta_\varepsilon \Rightarrow |f(x) - l| < \varepsilon]$$

(l_4) x_0 – finite, $l = +\infty$

$$\lim_{x \to x_0} f(x) = +\infty \iff$$

$$\iff [\forall \varepsilon\ \exists \delta_\varepsilon > 0\ \forall x \neq x_0,\ |x - x_0| < \delta_\varepsilon \Rightarrow f(x) > \varepsilon]$$

(l_5) $x_0 = +\infty$, $l = +\infty$

$$\lim_{x \to \infty} f(x) = +\infty \iff$$

$$\iff [\forall \varepsilon\ \exists \delta_\varepsilon\ \forall x > \delta_\varepsilon \Rightarrow f(x) > \varepsilon]$$

(l_6) $x_0 = -\infty$, $l = +\infty$

$$\lim_{x \to -\infty} f(x) = +\infty \iff$$

$$\iff [\forall \varepsilon\ \exists \delta_\varepsilon\ \forall x < \delta_\varepsilon \Rightarrow f(x) > \varepsilon]$$

(l_7) x_0 – finite, $l = -\infty$

$$\lim_{x \to x_0} f(x) = -\infty \iff$$

$$\iff [\forall \varepsilon\ \exists \delta_\varepsilon > 0\ \forall x \neq x_0,\ |x - x_0| < \delta_\varepsilon \Rightarrow f(x) < \varepsilon]$$

(l_8) $x_0 = +\infty$, $l = -\infty$

$$\lim_{x \to -\infty} f(x) = -\infty \iff$$

$$\iff [\forall \varepsilon\ \exists \delta_\varepsilon\ \forall x > \delta_\varepsilon \Rightarrow f(x) < \varepsilon]$$

$(l_p)\quad x_o = -\infty,\quad l = -\infty$

$$\lim_{x \to -\infty} f(x) = -\infty \iff$$

$$\iff \left[\, \forall \varepsilon\ \exists \delta_\varepsilon\ \forall\ x<\delta_\varepsilon \Rightarrow f(x)<\varepsilon \,\right]$$

Table 3.1

$x_o \backslash l$	l - finit	$l = +\infty$	$l = -\infty$
x_o finite	$V_{x_o} = (x_o-\delta, x_o+\delta)$ $V_l = (l-\varepsilon, l+\varepsilon)$	$V_{x_o} = (x_o-\delta, x_o+\delta)$ $V_l = (\varepsilon, \infty)$	$V_{x_o} = (x_o-\delta, x_o+\delta)$ $V_l = (-\infty, \varepsilon)$
	$x \in V_{x_o} \iff$ $\iff \lvert x-x_o\rvert < \delta$	$x \in V_{x_o} \iff$ $\iff \lvert x-x_o\rvert < \delta$	$x \in V_{x_o} \iff$ $\iff \lvert x-x_o\rvert < \delta$
	$f(x) \in V_l \iff$ $\iff \lvert f(x)-l\rvert < \varepsilon$	$f(x) \in V_l \iff$ $\iff f(x) > \varepsilon$	$f(x) \in V_l \iff$ $\iff f(x) < \varepsilon$
$x_o = +\infty$	$V_{x_o} = (\delta, \infty)$ $V_l = (l-\varepsilon, l+\varepsilon)$	$V_{x_o} = (\delta, \infty)$ $V_l = (\varepsilon, \infty)$	$V_{x_o} = (\delta, \infty)$ $V_l = (-\infty, \varepsilon)$
	$x \in V_{x_o} \iff$ $\iff x > \delta$	$x \in V_{x_o} \iff$ $\iff x > \delta$	$x \in V_{x_o} \iff$ $\iff x > \delta$
	$f(x) \in V_l \iff$ $\iff \lvert f(x)-l\rvert < \varepsilon$	$f(x) \in V_l \iff$ $\iff f(x) > \varepsilon$	$f(x) \in V_l \iff$ $\iff f(x) < \varepsilon$
$x_o = -\infty$	$V_{x_o} = (-\infty, \delta)$ $V_l = (l-\varepsilon, l+\varepsilon)$	$V_{x_o} = (-\infty, \delta)$ $V_l = (\varepsilon, \infty)$	$V_{x_o} = (-\infty, \delta)$ $V_l = (-\infty, \varepsilon)$
	$x \in V_{x_o} \iff$ $\iff x < \delta$	$x \in V_{x_o} \iff$ $\iff x < \delta$	$x \in V_{x_o} \iff$ $\iff x < \delta$
	$f(x) \in V_l \iff$ $\iff \lvert f(x)-l\rvert < \varepsilon$	$f(x) \in V_l \iff$ $\iff f(x) > \varepsilon$	$f(x) \in V_l \iff$ $\iff f(x) < \varepsilon$

Exercises

I. Using the definition, prove that:

1. $\lim_{x \to 5} (3x - 1) = 14$

2. $\lim_{x \to 1} \frac{2x + 3}{3x + 2} = 1$

3. $\lim_{x \to 2} \sqrt{x + 2} = 2$

4. $\lim_{x \to 1} \frac{1}{(x - 1)^2} = +\infty$

5. $\lim_{x \to 0} \frac{-1}{5x^2} = -\infty$

6. $\lim_{x \to a} \sin x = \sin a$

7. $\lim_{x \to a} \ln x = \ln a$

ANSWERS:

In all cases, we have $x_0 - finite$.

2. We go through the following stages:

a) we particularize the definition of limit

$$\lim_{x \to 1} \frac{2x + 3}{3x + 2} = 1 \iff$$

$$\iff [\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0\ \forall x \neq 1, |x - 1| < \delta_\varepsilon \Rightarrow$$

$$\Rightarrow \left| \frac{2x + 3}{3x + 2} - 1 \right| < \varepsilon]$$

(taking into account the form of the vicinity).

b) we consider $\varepsilon > 0$, irrespective; we therefore search for δ_ε.

c) we make calculation in the expression $|f(x) - l|$, highlighting the module $|x - x_0|$.

$$|f(x) - l| = \left| \frac{2x + 3}{3x + 2} - 1 \right| = \frac{|x - 1|}{|3x + 2|}.$$

d) we increase (decrease) the obtained expression, keeping in mind we are ony interested in those values of x for which $|x - x_o| < \delta_\varepsilon$.

So: $\frac{|x-1|}{|3x+2|} < \frac{\delta_\varepsilon}{|3x+2|}$.

e) if the expression that depends on x is bounded (continuous) in a vicinity of x_o, we can further increase, to eliminate x. We have:

$$\frac{\delta_\varepsilon}{|3x+2|} = \delta_\varepsilon \frac{1}{|3x+2|} < \delta_\varepsilon \frac{1}{2}$$

because in the interval $[0,2]$, for example, that is a vicinity of $x_o = 1$, the function $g(x) = \frac{1}{|3x+2|}$ is continuous, and therefore bounded: $g(x) \in [\frac{1}{5}, \frac{1}{2}]$, if $x \in [0,2]$.

f) we determine δ_ε, setting the condition that the expression we have reached (and that doesn't depend on x, just on δ_ε) be smaller than ε:

$$\delta_\varepsilon \frac{1}{2} < \varepsilon \Rightarrow \delta_\varepsilon < 2\varepsilon$$

Any δ_ε that fulfills this condition is satisfactory. We can, for example, choose $\delta_\varepsilon \frac{3}{2}\varepsilon$ or $\delta_\varepsilon = \varepsilon$ etc.

In order to use the increase made to function g, we must also have $\delta_\varepsilon \leq 1$, so, actually $\delta_\varepsilon = \min(1, \frac{3}{2}\varepsilon)$ (for example).

It follows that:

$$|f(x) - l| = \frac{|x-1|}{|3x+2|} < \frac{\delta_\varepsilon}{|3x+2|} <$$

$$< \frac{\delta_\varepsilon}{2} = \min(1, \frac{3}{2}\varepsilon) < \varepsilon$$

3. a) $\lim_{x \to 2} \sqrt{x+2} = 2 \Leftrightarrow$

$\Leftrightarrow [\ \forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 \ \forall x \neq 2,\ |x - 2| < \delta_\varepsilon \Rightarrow$

$$\Rightarrow \left| \sqrt{x+2} - 2 \right| < \varepsilon \,]$$

b) let $\varepsilon > 0$; we search for δ_ε

c) $$\left| \sqrt{x+2} - 2 \right| = \frac{|x-2|}{\left| \sqrt{x+2} + 2 \right|}$$

d) $$\frac{|x-2|}{\sqrt{x+2}+2} < \frac{\delta_\varepsilon}{\sqrt{x+2}+2}$$

e) $$\frac{\delta_\varepsilon}{\sqrt{x+2}+2} = \delta_\varepsilon \frac{1}{\sqrt{x+2}+2} < \delta_\varepsilon \frac{1}{3},$$

because in the interval $[1,3]$, for example, the function:

$$g(x) = \frac{1}{\sqrt{x+2}+2}$$

is bounded by:

$$\frac{1}{\sqrt{5}+2} \text{ and } \frac{1}{\sqrt{3}+2}$$

f) we determine δ_ε from the condition: $\frac{\delta_\varepsilon}{3} < \varepsilon$. We obtain $\delta_\varepsilon < 3\varepsilon$, so we can, for example, take $\delta_\varepsilon = \varepsilon$. But in order for the increase we have made to function g to be valid, we must also have $\delta_\varepsilon \leq 1$, so, actually $\delta_\varepsilon = \min(1, \varepsilon)$.

It follows that:

$$\frac{|x+2|}{\sqrt{x+2}+2} < \frac{\delta_\varepsilon}{\sqrt{x+2}+2} <$$

$$< \frac{\delta_\varepsilon}{3} = \frac{\min(1, \varepsilon)}{3} < \varepsilon$$

so the condition from the definition of the limit is in this case, also, fulfilled.

4) a) $\lim\limits_{x \to 1} \dfrac{1}{(x-1)^2} = \infty \quad \Leftrightarrow$

$\Leftrightarrow \left[\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 \ \forall x \neq 1, |x-1| < \delta_\varepsilon \Rightarrow\right.$

$\left.\Rightarrow \dfrac{1}{(x-1)^2} > \varepsilon\right]$

b) let ε any (not necessarily positive, because $l = +\infty$, so $V_l = (\varepsilon, +\infty)$only makes sense for any ε, not just for ε positive).

c) $\dfrac{1}{(x-1)^2} = \dfrac{1}{|x-1|^2}$

d) $\dfrac{1}{|x-1|^2} < \dfrac{1}{\delta_\varepsilon^2}$ (**because we only consider those x for which**, $|x-1| < \delta$).

f) $\dfrac{1}{\delta_\varepsilon^2} > \varepsilon \Rightarrow \delta_\varepsilon^2 < \dfrac{1}{\varepsilon}$,

so we can take $\delta_\varepsilon = \frac{1}{2\sqrt{\varepsilon}}$

Then:

$$\frac{1}{(x-1)^2} = \frac{1}{|x-1|^2} >$$

$$> \frac{1}{\delta_\varepsilon^2} = \frac{1}{\frac{1}{4\varepsilon}} = 4\varepsilon > \varepsilon$$

II. Using the definition, prove that:

1. $\lim\limits_{x \to \infty} \dfrac{3x+1}{2x-1} = \dfrac{3}{2}$

2. $\lim\limits_{x \to \infty} \dfrac{x^2+1}{x-100} = \infty$

ANSWERS:

1. a) $\lim\limits_{x \to \infty} \dfrac{3x+1}{2x-1} = \dfrac{3}{2} \quad \Leftrightarrow$

$\Leftrightarrow \quad \left[\forall \varepsilon > 0 \ \exists \delta_\varepsilon \ \forall x > \delta_\varepsilon \Rightarrow\right.$

$$\Rightarrow \quad \left| \frac{3x+1}{2x-1} - \frac{3}{2} \right| < \varepsilon \,]$$

b) let $\varepsilon > 0$ any; we search for δ_ε.

c) because x_0 isn't finite anymore, we cannot highlight $|x - x_0|$. We will proceed differently: we consider the inequality $|f(x) - l| < \varepsilon$ as an inequation in the unknown x (using also the fact that $x \to \infty$, so, we can consider $|2x+1| = 2x+1$). We have:

$$|f(x) - l| < \varepsilon \Leftrightarrow \left| \frac{3x+1}{2x-1} - \frac{3}{2} \right| < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2|2x+1|} < \varepsilon \Leftrightarrow \frac{1}{2(2x+1)} <$$

$$< \varepsilon \left[\text{for } x > -\frac{1}{2} \right] \Leftrightarrow 2x+1 > \frac{1}{2\varepsilon} \Leftrightarrow$$

$$\Leftrightarrow x > \frac{1-2\varepsilon}{4\varepsilon}$$

So for $\delta_\varepsilon = \max\left(-\frac{1}{2}, \frac{1-2\varepsilon}{4\varepsilon}\right)$ and

$x > \delta_\varepsilon$, we have:

$$\left| \frac{3x+1}{2x-1} - \frac{3}{2} \right| < \varepsilon .$$

2. a) $\lim\limits_{x \to \infty} \dfrac{x^2+1}{x-100} = \infty \Leftrightarrow$

$$\Leftrightarrow \left[\forall \varepsilon \;\; \exists \delta_\varepsilon \;\; \forall x > \delta_\varepsilon \;\; \frac{x^2+1}{x-100} > 100 \right]$$

b) let $\varepsilon > 0$ any; we determine δ_ε.

c) we express x from the inequality $f(x) > \varepsilon$:

$\dfrac{x^2+1}{x-100} > \varepsilon \Leftrightarrow$ (we can assume $x > 100$ because $x \longrightarrow \infty$) $\Leftrightarrow \; x^2 + 1 - \varepsilon x + 100\varepsilon > 0.$

If x_1 and x_2 are the roots of the attached second degree equation and we assume $x_1 < x_2$, we have $f(x) > \varepsilon \Leftrightarrow x > \max(100, x_2)$.

So we can take $\delta_\varepsilon = \max(100, x_2)$.

The limits of sequences

The definition of a sequence's limit is deduced from the definition of a function's limit, making the mentioned particularizations:

(1) x is replaced with n,

(2) $f(x)$ is replaced with a_n,

(3) x_o is replaced with n_o.

Moreover, we have to observe that $n_o = -\infty$ and finite n_o don't make sense (for example $n \to 3$ doesn't make sense, because n, being a natural number, cannot come no matter how close to 3).

So, from the nine forms of a function's limit, only 3 are particularized: the ones corresponding to $x_o = +\infty$.

We therefore have:

(l_{10}) $l - finite$ (we transpose the definition (l_2))

$$\lim_{n \to \infty} a_n = l \iff$$

$$\iff [\forall \varepsilon > 0 \ \exists \delta_\varepsilon \ \forall n > \delta_\varepsilon \ |a_n - l| < \varepsilon]$$

Because in the inequality $n > \delta_\varepsilon$, n is a natural number, we can consider δ_ε as a natural number also. To highlight this, we will write n_ε instead of δ_ε. We thus have:

$$\lim_{n \to \infty} a_n = l \iff$$

$$\iff [\forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N} \ \forall n > n_\varepsilon \Rightarrow |a_n - l| < \varepsilon]$$

(l_{11}) $l = +\infty$ (we transpose the definition (l_5))

$$\lim_{n\to\infty} a_n = \infty \iff$$

$$\iff [\forall \varepsilon>0 \ \exists n_\varepsilon \in \mathbb{N} \ \forall n>n_\varepsilon \Rightarrow a_n > \varepsilon]$$

(l_{12}) $l = -\infty$ (we transpose the definition (l_8))

$$\lim_{n\to\infty} a_n = -\infty \iff$$

$$\iff [\forall \varepsilon \ \exists n_\varepsilon \in \mathbb{N} \ \forall n>n_\varepsilon \Rightarrow a_n < \varepsilon]$$

Examples

I. Using the definition, prove that:

1. $\lim_{n\to\infty} \frac{3n-1}{3n+1} = 1$

2. $\lim_{n\to\infty} \frac{1}{n^2+n+1} = 0$

3. $\lim_{n\to\infty} \frac{3}{2^n+3} = 0$

4. $\lim_{n\to\infty} \sqrt{n^2+n+1} = \infty$

ANSWERS:

1. We adapt the corresponding stages for functions' limits, in the case of $x_0 = +\infty$.

a) $\lim_{n\to\infty} \frac{3n-1}{3n+1} = 1 \iff$

$$\iff [\forall \varepsilon>0 \ \exists n_\varepsilon \in \mathbb{N} \ \forall n>n_\varepsilon \left|\frac{3n-1}{3n+1} - 1\right| < \varepsilon]$$

b) let $\varepsilon > 0$ any; we determine $n_\varepsilon \in \mathbb{N}$.

c) we consider the inequality: $|a_n - l| < \varepsilon$ as an inequation with the unknown n:

$$\left|\frac{3n-1}{3n+1}-1\right|<\varepsilon \Leftrightarrow$$

$$\Leftrightarrow \frac{2}{3n+1}<\varepsilon \Leftrightarrow n>\frac{\frac{2}{\varepsilon}-1}{3}$$

so, $n_\varepsilon = \left[\frac{2-\varepsilon}{3\varepsilon}\right]+1$

$$2.\ \text{a)}\ \lim_{n\to\infty}\frac{1}{n^2+n+1}=0 \Leftrightarrow$$

$$\Leftrightarrow \left[\forall\, \varepsilon>0\ \exists\, n_\varepsilon\in\mathbb{N}\ \forall\, n>n_\varepsilon \left|\frac{1}{n^2+n+1}\right|<\varepsilon\right]$$

b) let $\varepsilon > 0$ any; we determine $n_\varepsilon \in \mathbb{N}$ as follows:

$$\text{c)}\ \frac{1}{n^2+n+1}<\varepsilon \Leftrightarrow$$

$$\Leftrightarrow \varepsilon n^2+\varepsilon n+\varepsilon-1>0$$

Let n_1 and n_2 represent the solutions to this second degree equation $(n_1 < n_2)$. Then, for $n_1 > n_2$, the required inequality is satisfied. So, $n_\varepsilon = [n_2] + 1$.

Calculation methods for the limits of functions and sequences

Let's now look at the most common (for the high school workbook level) calculation methods of the limits of functions and sequences.

Joint methods for sequences and functions

1. Definition

In the first chapter, we have shown how to use the definition to demonstrate the limits of both functions and sequences. We will add to what has already been said, the following set of exercises:

I. Using the definition, find out if there limits to the following sequences and functions:

1. $a_n = \dfrac{\sqrt{n^2 - 100}}{n}$

2. $a_n = \sqrt{n^2 - 100}$

3. $a_n = \dfrac{\sqrt{n^2 - 100}}{700 - n}$

4. $f(x) = \sqrt{x}$ – in $x_0 = 2$ şi $x_0 = a$

5. $f(x) = \cos x$ – in $x_0 = \dfrac{\pi}{2}$ şi $x_0 = a$

6. $f(x) = 5^{x^2+1}$ – in $x_0 = 1$ şi $x_0 = a$

2. Giving the forced common factor

The method is frequently used to eliminate the indeterminations of type: $\frac{\infty}{\infty}$, $\infty - \infty$, $\frac{0}{0}$. By removing the forced common factor we aim at obtaining as many expressions as possible that tend towards zero. For this, we commonly employ the following three limits:

$$(1)\quad \lim_{\substack{x \to 0 \\ x > 0}} x^{\alpha} = \begin{cases} 0 & \text{if } \alpha > 0 \\ 1 & \text{if } \alpha = 0 \\ \infty & \text{if } \alpha < 0 \end{cases}$$

$$(2)\quad \lim_{x \to \infty} x^{\alpha} = \begin{cases} \infty & \text{if } \alpha > 0 \\ 1 & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha < 0 \end{cases}$$

$$(3)\quad \lim_{n \to \infty} x^{n} = \begin{cases} 0 & \text{if } x \in (-1, 1) \\ 1 & \text{if } x = 1 \\ \infty & \text{if } x > 1 \end{cases}$$

In order to have expressions that tend towards zero we aim therefore to:

- obtain as many terms as possible with the form x^{α}, with $\alpha < 0$, if $x \to \infty$.
- obtain as many terms as possible with the form x^{α}, with $\alpha > 0$, if $x \to 0$.

Examples

I. Calculate:

1. $$\lim_{x \to \infty} \frac{3x^2 - x + 2}{x^2 + x + 5}$$

2. $$\lim_{n \to \infty} \frac{7^{n+3} + 9^n}{7^{n+2} - 9^{n+1}}$$

3. $$\lim_{n \to \infty} \frac{2^n + 2 \cdot 3^n + 3 \cdot 5^n}{3^n + 3 \cdot 4^n + 4 \cdot 5^n}$$

4. $$\lim_{n \to \infty} \frac{2^n + 3^n + x^n}{3^n + 4^n}, \quad x > 0$$

5. $$\lim_{n \to \infty} \frac{2 \cdot a^n + b^n}{3 \cdot a^n + 4 \cdot b^n}$$

6. $$\lim_{n \to \infty} \frac{(n + 2)! - (n + 1)!}{(n + 2)! + (n + 1)!}$$

7. $$\lim_{n \to \infty} \frac{\sqrt{n^3 - 2 \cdot n^2 - 2} + \sqrt[3]{n^4 + 1}}{\sqrt[4]{n^6 + 6 \cdot n^5 + 2} + \sqrt[5]{n^7 + 3 \cdot n^3 + 1}}$$

8. $$\lim_{\substack{x \to 0 \\ x > 0}} \frac{\sqrt{x + 2 \cdot \sqrt[3]{x}}}{\sqrt[3]{2x - \sqrt{x}}}$$

9. $$\lim_{x \to \infty} \frac{\ln(x^2 + x + 1)}{\ln(x^{10} + x + 2)}$$

10. $$\lim_{\substack{x \to 0 \\ x > 0}} \frac{\sqrt[m]{x + 3 \cdot \sqrt[n]{x}}}{\sqrt[n]{3x + \sqrt[m]{x}}}$$

11. $\lim\limits_{x \to \infty} \dfrac{(x+1)\cdot\ldots\cdot(x^n+1)}{\left[(nx)^n+1\right]^{\frac{n+1}{2}}}$

12. $\lim\limits_{x \to 0} \dfrac{\ln|x|}{1+\ln|x|}$

13. $\lim\limits_{x \to \infty} \dfrac{\ln(x+e^x)}{\ln(x^2+e^{2x})}$

ANSWERS:

2. Given the common factor 9^n we obtain terms that have the form x^n, with sub unitary x.

9. $\dfrac{\ln(x^2+x+1)}{\ln(x^{10}+x+2)} =$

$$= \frac{\ln x^2\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}{\ln x^{10}\left(1+\frac{1}{x^9}+\frac{1}{x^{10}}\right)} =$$

$$= \frac{2\ln x+\ln\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}{10\ln x+\ln\left(1+\frac{1}{x^9}+\frac{1}{x^{10}}\right)},$$

expression that tends to $\frac{1}{5}$ when x tends to infinity.

II. Trace the graphics of the functions:

1. $f(x) = \lim\limits_{n \to \infty} \dfrac{2^n}{2^n+x^n}, \; x > 0$

2. $f(x) = \lim\limits_{n \to \infty} \dfrac{x^{2n+2}}{2^{2n}+x^{2n}}$

3. $f(x) = \lim\limits_{n \to \infty} \dfrac{1+x+\ldots+x^n}{1+x^n}$

4. $f(x) = \lim\limits_{n \to \infty} \sqrt{1 + x^n}\ ,\ x > -1$

5. $f(x) = \lim\limits_{n \to \infty} \dfrac{\ln(2^n + x^n)}{n}$

6. $f(x) = \lim\limits_{n \to \infty} (x - 1) \cdot \operatorname{arctg} x^n$

7. $f(x) = \lim\limits_{n \to \infty} \dfrac{x^{n+1} - 2 \cdot \ln^{n+1} x}{x^n + 3 \cdot \ln^n x}$

8. For any rational function, non-null, R with real coefficients, we have:

$$\lim_{x \to \infty} \frac{R(x)}{R(x+1)} = 1$$ **(workbook grade XII)**

3. Amplification with the conjugate

It is used to eliminate the indeterminations that contain radicals. If the indetermination comes from an expression with the form:

$$\sqrt[p]{u(x)} - \sqrt[p]{v(x)}$$

than we amplify with:

$$\sqrt[p]{u^{p-1}(x)} + \sqrt{u^{p-2}(x) \cdot v(x)} + \ldots + \sqrt{v^{p-1}(x)} \qquad (3.1)$$

with the purpose of eliminating the radicals from the initial expression. The sum from (3.1) is called a conjugate of order p.

If the indetermination comes from an expression with the form:

$$\sqrt[p]{u(x)} + \sqrt[p]{v(x)}$$

with p – uneven number, the conjugate is:

$$\sqrt[p]{u^{p-1}(x)} - \sqrt[p]{u^{p-2}(x) \cdot v(x)} + \ldots + \sqrt[p]{v^{p-1}(x)}\ .$$

An example of application for this formula is exercise 5, below.

Exercises

$$1.\ \lim_{x \to \infty} \left(\sqrt[3]{x^3 + x^2 + 1} - \sqrt[3]{x^3 - x^2 + 1} \right)$$

$$2.\ \lim_{x \to \infty} \frac{\sqrt{x^2 + 1} + \sqrt{x^2 - 2x + 3}}{x - 3}$$

$$3.\ \lim_{x \to 0} \frac{\sqrt[n]{1 + x} - \sqrt[n]{1 - x}}{x}$$

$$4.\ \lim_{x \to 0} \frac{\sqrt[m]{1 + x} + \sqrt{1 - x}}{x}$$

$$5.\ \lim_{x \to -1} \frac{1 + \sqrt[3]{x}}{1 + \sqrt[5]{x}}$$

$$6.\ \lim_{n \to \infty} n^{\frac{m-p}{mp}} \frac{\sqrt[m]{n + 1} - \sqrt[m]{n - 1}}{\sqrt[p]{n + 1} - \sqrt[p]{n - 1}}$$

$$7.\ \lim_{n \to \infty} \left(a_0 \sqrt[3]{n} + a_1 \sqrt[3]{n + 1} + \right.$$

$$\left. + \ldots + a_k \sqrt[3]{n + k} \right), \text{ with } a_0 + a_1 + \ldots + a_k = 0$$

ANSWERS:

7. We replace, for example $a_0 = -a_1 - a_2 - \cdots - a_k$, in the given sequence and we calculate k limits of the form:

$$\lim_{n \to \infty} a_i \left(\sqrt[3]{n} - \sqrt[3]{n + i} \right)$$

4. Using fundamental limits

Hereinafter, we will name fundamental limits the following limits:

$$\text{(a)} \qquad \lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = 1$$

(b) $$\lim_{\alpha \to 0} (1 + \alpha)^{\frac{1}{\alpha}} = e$$

(c) $$\lim_{\alpha \to 0} \frac{a^{\alpha} - 1}{\alpha} = \ln a$$

OBSERVATION:

From (a) we deduce that:

$$\lim_{\alpha \to 0} \frac{\arcsin \alpha}{\alpha} = 1 ; \quad \lim_{\alpha \to 0} \frac{\operatorname{tg} \alpha}{\alpha} = 1 ;$$

$$\lim_{\alpha \to 0} \frac{\operatorname{arctg} \alpha}{\alpha} = 1$$

From (b) we deduce that:

$$\lim_{\alpha \to \infty} \left(1 + \frac{1}{\alpha} \right)^{\alpha} = e$$

From (c) we deduce that:

$$\lim_{\alpha \to 0} \frac{\ln(1 + \alpha)}{\alpha} = 1 ; \quad \lim_{\alpha \to \alpha_0} \frac{a^{\alpha} - a^{\alpha_0}}{\alpha - \alpha_0} = a^{\alpha} \cdot \ln a$$

Exercises

I. Calculate:

1. $\lim_{x \to 0} \frac{\sin 3x}{\sin 7x}$

2. $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

3. $\lim_{x \to 0} \frac{1 - \cos kx}{x^2}$

4. $\lim_{x \to 0} \frac{1 - \cos^3 x}{x^2}$

5. $\lim_{x \to \pi} \frac{\sin 3x}{\sin 8x}$

6. $\lim\limits_{x \to 0} \dfrac{\sin x - \operatorname{tg} x}{x^2 \operatorname{tg} x}$

7. $\lim\limits_{x \to 0} \dfrac{1 - \cos x \cdot \cos 2x}{x^2}$

8. $\lim\limits_{x \to 0} \dfrac{1 - \cos x \cdot \ldots \cdot \cos nx}{x^2}$

9. $\lim\limits_{x \to 1} \dfrac{\cos \pi x + 1}{(x - 1)^2}$

10. $\lim\limits_{x \to \pi} \dfrac{\sin mx}{\sin nx}$

11. $\lim\limits_{x \to 0} \dfrac{x^m - \sin^n x}{x^{n+2}}$ **for different values of** $n \in \mathbb{N}$

12. $\lim\limits_{x \to 0} \dfrac{\arcsin x - \operatorname{arctg} x}{x^3}$

13. $\lim\limits_{x \to 0} \dfrac{\ln(1 + x) - \ln(1 - x)}{\operatorname{arctg}(1 + x) - \operatorname{arctg}(1 - x)}$

ANSWERS:

12. Noting with $\alpha = \arcsin x - arctg\ x$, we observe that α tends to zero when x tends to zero, so we aim to obtain $\frac{\alpha}{\sin \alpha}$. To accomplish this we amplify with:

$$\sin \alpha = \sin(\arcsin x - \operatorname{arctg} x) =$$
$$= \sin(\arcsin x)\cos(\operatorname{arctg} x) -$$
$$- \sin(\operatorname{arctg} x)\cos(\arcsin x).$$

Noting with:

$u = \arcsin x$, $v = \operatorname{arctg} x$,

it follows:

$x = \sin u$, so $\cos u = \sqrt{1 - x^2}$

$x = \operatorname{tg} v$, so

$$\cos v = \frac{1}{\sqrt{1 + x^2}}, \sin v = \frac{1}{\sqrt{1 + x^2}}$$

II. Calculate:

1. $\lim\limits_{x \to \infty} \left(\frac{x^2 + 1}{x^2 - 2} \right)^{x^2}$

2. $\lim\limits_{x \to 0} \left(\frac{\cos x}{\cos 2x} \right)^{\frac{1}{x^2}}$

3. $\lim\limits_{n \to \infty} \operatorname{tg}^n \frac{\pi}{4}$

4. $\lim\limits_{\to 0} \left(1 + x^2 \right)^{\operatorname{ctg}^2 x}$

5. $\lim\limits_{x \to 0} \left(\frac{a^{x^2} + b^{x^2}}{a^x + b^x} \right)^{\frac{1}{x}}$

6. $\lim\limits_{x \to -\infty} \frac{\ln(1 + 3^x)}{\ln(1 + 2^x)}$

7. $\lim\limits_{x \to e} \left(\ln x \right)^{\frac{1}{x^2 - 3ex + 2e^2}}$

8. $\lim\limits_{n \to \infty} \frac{3^{3n}}{\left(3 + \frac{1}{n} \right)^{3n}}$

ANSWERS:

6. In the limits where there are indeterminations with logarithms, we aim to permute the limit with the logarithm:

$$\lim_{x \to -\infty} \frac{1}{\ln(1 + 2^x)} \cdot \ln(1 + 3^x) =$$

$$= \lim_{x \to -\infty} \ln\left(1 + 3^x\right)^{\frac{1}{\ln(1 + 2^x)}} =$$

$$= \ln \lim_{x \to -\infty}\left(1 + 3^x\right)^{\frac{1}{\ln(1 + 2^x)}} =$$

$$= \ln \lim_{x \to -\infty}\left[\left(1 + 3^x\right)^{\frac{1}{3^x}}\right]^{\frac{1}{\ln(1 + 2^x)}} =$$

$$= \lim_{x \to -\infty} \frac{3^x}{\ln(1 + 2^x)} =$$

= (we permute again the limit with the logarithm) =

$$= \frac{1}{\lim\limits_{x \to -\infty} \frac{1}{3^x} \cdot \ln(1 + 2^x)} =$$

$$= \frac{1}{\lim\limits_{x \to -\infty} \ln\left(1 + 2^x\right)^{\frac{1}{3^x}}} = \frac{1}{\ln \lim\limits_{x \to -\infty}\left(1 + 2^x\right)^{\frac{1}{3^x}}} =$$

$$= \frac{1}{\ln \lim\limits_{x \to -\infty}\left[\left(1 + 2^x\right)^{\frac{1}{2^x}}\right]^{\frac{2^x}{3^x}}} = \frac{1}{\infty} = 0$$

III. Calculate:

1. $\lim\limits_{x \to a} \dfrac{a^x - a^a}{x - a}$

2. $\lim\limits_{x \to 0} \dfrac{2^{tg\ x} - 1}{x}$

3. $\lim\limits_{x \to \pi/4} \dfrac{9^{\sin x} - 3^{\sqrt{2}}}{x - \frac{\pi}{4}}$

4. $\lim\limits_{n \to \infty} n\left(\sqrt[n]{2} - 1\right)$

5. $\lim\limits_{n \to \infty} n\left(\cos\dfrac{1}{n} - 1\right)$

6. $\lim\limits_{n \to \infty} n \cdot \arcsin\dfrac{1}{n}$

7. $\lim\limits_{n \to \infty} n\left(\sqrt[3]{\cos\dfrac{1}{n}} - 1\right)$

8. $\lim\limits_{n \to \infty} n\left(\sqrt[n]{a} - 1\right)$

9. $\lim\limits_{n \to \infty} n\left(\sqrt[n]{a_1} + \sqrt[n]{a_2} + \ldots + \sqrt[n]{a_p} - p\right)$

ANSWERS:

1. $\lim\limits_{x \to a} \dfrac{a^x - a^a}{x - a} = \lim\limits_{x \to a} \dfrac{a^a\left(a^{x-a} - 1\right)}{x - a}$

and noting $x - a = \alpha$, we obtain:

$$l = a^a \lim_{\alpha \to 0} \frac{a^\alpha - 1}{\alpha} = a^a \ln a .$$

4. We consider the function:

$$f(x) = x\left(2^{\frac{1}{x}} - 1\right)$$

obtained from a_n, by replacing n with x. We calculate:

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{2^{\frac{1}{x}} - 1}{\frac{1}{x}} = \ln 2.$$

According to the criterion with sequences (see *Method 10, point c*), we also have: $\lim\limits_{n \to \infty} a_n = \ln 2$.

5. Something that is bounded multiplied with something that tends to zero, tends to zero

(A) If $(a_n)_{n\in\mathbb{N}}$ is bounded and $\lim_{n\to\infty} b_n = 0$, then:

$$\lim_{n\to\infty} a_n . b_n = 0$$

(B) If f is bounded in a vicinity of x_0 and $\lim_{x\to x_0} g(x) = 0$, then $\lim_{x\to x_0} f(x).g(x) = 0$.

Examples

1. $$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

because $(-1)^n$ is bounded by -1 and 1 and $\frac{1}{n}$ tends to zero;

2. $$\lim_{x\to\infty} \frac{1-\cos x}{x^2} = 0$$

because $1-\cos x$ is bounded by 0 and 2 and $\frac{1}{x^2}$ tends to zero;

3. $$\lim_{x\to\infty} \frac{\sin x}{x} = 0;$$

4. $$\lim_{n\to\infty} \frac{a_n}{n},$$

where a_n is the n^th decimal fraction of the number π;

5. $$\lim_{n\to\infty} \frac{\beta_n}{2n^2+1},$$

where β_n is the approximation with n exact decimal fractions of the number e.

6. $$\lim_{n\to\infty} \frac{2^n+(-2)^n}{n-2^n};$$

7. $$\lim_{n\to\infty} \frac{\operatorname{sgn}(n^2-3n+2)}{e^{n+1}};$$

8. $\lim\limits_{n \to \infty} \dfrac{\sin(n^2 + 3n + 5)}{n^2 + 3n + 5}$;

9. $\lim\limits_{n \to \infty} \dfrac{a \cdot ctg(n \cdot x)}{nx^2 + 2}$;

10. $\lim\limits_{x \to 0} x \cdot arctg \dfrac{1}{x}$;

11. $\lim\limits_{n \to \infty} \dfrac{n \cdot \sin(n!)}{n^2 + 1}$;

12. $\lim\limits_{x \to \infty} \dfrac{x - [x]}{x}$;

13. $\lim\limits_{x \to 0} \dfrac{(-1)^{[x]} \cdot \sin(x^2)}{x}$;

INDICATIONS:

4. $(a_n)_{n\in\mathbb{N}}$ is bounded by 0 and 9, being composed of numbers;

5. $(\beta_n)_{n\in\mathbb{N}}$ is bounded by 2 and 3;

6. The majoring and minoring method

(A) If functions g and h exist, so that in a vicinity of x_o we have:

$g(x) \le f(x) \le h(x)$ and $\lim\limits_{x \to x_o} g(x) = \lim\limits_{x \to x_o} h(x) = l$

then:

$$\lim_{x \to x_o} f(x) = l.$$

Schematically:

$$\begin{array}{ccccc} g(x) & \le & f(x) & \le & h(x) \\ \downarrow & & \Downarrow & & \downarrow \\ \to & & l & & \leftarrow \end{array}$$

(B) If the sequences $(b)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ exist so that: $b_n \le a_n \le c_n$, starting from the rank n_0 and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = l$, then: $\lim_{n\to\infty} a_n = l$.

Schematically:

$$\begin{tikzcd} b_n \arrow[r, phantom, "\le"] \arrow[dr] & a_n \arrow[r, phantom, "\le"] \arrow[d, Rightarrow] & c_n \arrow[dl] \\ & l & \end{tikzcd}$$

OBSERVATION:

If $l = +\infty$, we can eliminate h $((c_n)_{n\in\mathbb{N}}\ respectively)$, and if $l = -\infty$, we can eliminate g $((b_n)_{n\in\mathbb{N}}\ respectively)$.

This method is harder to apply, due to the majorings and minorings it presupposes. These have to lead to expressions, as least different as the initial form as possible, not to modify the limit.

We mention a few methods, among the most frequently used, to obtain the sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$.

Examples

1. We put the smallest (biggest) term of the sequence $(a_n)_{n\in\mathbb{N}}$, instead of all the terms.

a) $a_n = \dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \ldots + \dfrac{1}{\sqrt{n^2+n}}$

b) $a_n = \dfrac{1}{\sqrt[p]{n^p+1}} + \dfrac{1}{\sqrt[p]{n^p+2}} + \ldots + \dfrac{1}{\sqrt[p]{n^p+n}}$

ANSWERS:

a) In the sum that expresses the sequence $(a_n)_{n\in\mathbb{N}}$, there are: the smallest term $\frac{1}{\sqrt{n^2+n}}$ and the biggest term $\frac{1}{\sqrt{n^2+1}}$. By applying *Method 1*, we have:

$$a_n \le \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \dots + \frac{1}{\sqrt{n^2+1}} = \frac{n}{\sqrt{n^2+1}} \longrightarrow 1$$

$$a_n \ge \frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+n}} + \dots + \frac{1}{\sqrt{n^2+n}} = \frac{n}{\sqrt{n^2+n}} \longrightarrow 1$$

so $a_n \longrightarrow 1$.

2. We minorate (majorate) the terms that make up the sequence a_n with the same quantity.

a) $a_n = \dfrac{\sin 1}{n^2+1} + \dfrac{\sin 2}{n^2+2} + \dots + \dfrac{\sin n}{n^2+n}$

b) $a_n = \dfrac{\sin x}{n^2+1} + \dfrac{\sin 2x}{n^2+2} + \dots + \dfrac{\sin nx}{n^2+n}$

ANSWERS:

a) This time there isn't a smallest (biggest) term in the sum that makes up the sequence a_n. This is due to the fact that the function $f(x) = \sin x$ isn't monotonic. Keeping into consideration that we have: $-1 \le \sin x \le 1$, we can majorate, replacing $\sin x$ with 1, all over. We obtain:

$$a_n \le \frac{1}{n^2+1} + \frac{1}{n^2+1} + \dots + \frac{1}{n^2+1} = \frac{1}{n^2+1} \longrightarrow 0$$

Analogously, we can minorate, replacing $\sin x$ with -1, then, by applying the first method it follows that:

$$a_n \ge \frac{-1}{n^2+n} + \frac{-1}{n^2+n} + \dots + \frac{-1}{n^2+n} = \frac{-n}{n^2+n} \longrightarrow 0$$

so $a_n \longrightarrow 0$.

3. We minorate (majorate), eliminating the last terms that make up the sequence a_n.

a) $a_n = \sqrt[n]{1^p + 2^p + \dots + n^p}$

b) $a_n = \sqrt[n]{1^n + 2^n + \dots + n^n}$

ANSWERS:

b) $a_n \geq \sqrt[n]{n^n} = n \longrightarrow \infty$, so $a_n \longrightarrow \infty$

Exercises

I. Calculate the limits of the sequences:

1. $a_n = \sum_{k=1}^{n} \left(\sqrt{1 + \frac{k}{n^2}} - 1 \right)$

2. $a_n = \sum_{k=1}^{n} \left(\sqrt{1 + \frac{k^2}{n^3}} - 1 \right)$

3. $a_n = \sum_{k=1}^{n} \operatorname{tg} \frac{2k}{n^2}$

4. $a_n = \sqrt{b_1^2 + b_2^2 + \dots + b_k^2}$,

with $1 < b_1 < b_2 < \dots < b_k$

5. $a_n = \sqrt{b_1^n + b_2^n + \dots + b_k^n}$,

with $b_i > 0$, $i \in \overline{1,k}$

ANSWERS:

$$1.\quad \sum_{k=1}^{n}\left[\sqrt{1+\frac{k}{n^2}}\;\right] = \sum_{k=1}^{n}\frac{\frac{k}{n^2}}{\sqrt{1+\frac{k}{n^2}}} =$$

$$= \frac{1}{n^2}\sum_{k=1}^{n}\frac{k}{\sqrt{1+\frac{k}{n^2}}}\,.$$

We majorate using the smallest and the biggest denominator. Thus, we obtain common denominators for the sum.

$$3.\quad tg\,\frac{2k}{n^2} = \frac{\sin\frac{2k}{n^2}}{\cos\frac{2k}{n^2}}$$

and we make majorings and minorings using the biggest and the smallest denominator.

7. Exercises that feature the integer part

The most common methods to solve them is:

(A) The minoring and majoring method using the double inequality:

$$a - 1 < [\,a\,] \le a \qquad (3.2)$$

that helps to encase the function (the sequence) whose limit we have to calculate, between two functions (sequences) that do not contain the integer part.

Using this method when dealing with functions, most of the times, we don't obtain the limit directly, instead we use the lateral limits.

Example: For the calculation of the limit:

$$\lim_{x \to -2}\frac{x^2+2x}{7}\left[\frac{5}{x^2-x-6}\right]$$

we proceed as follows:

(a) We use the inequalities (3.2) to obtain expressions that do not contain the integer part:

$$\frac{5}{x^2 - x - 6} - 1 < \left[\frac{5}{x^2 - x - 6}\right] \leq \frac{5}{x^2 - x - 6}$$

(b) To obtain the function whose limit is required, we amplify the previous inequalities with $\frac{x^2+2x}{7}$, keeping into consideration that:

- on the left of -2 we have: $\frac{x^2+2x}{7} > 0$, so:

$$\frac{x^2+2x}{7}\left(\frac{5}{x^2-x-6} - 1\right) < \frac{x^2+2x}{7}\left[\frac{5}{x^2-x-6}\right] <$$
$$< \frac{x^2+2x}{7} \cdot \frac{5}{x^2-x-6}$$

- on the right of -2 we have: $\frac{x^2+2x}{7} < 0$, so:

$$\frac{x^2+2x}{7}\left(\frac{5}{x^2-x-6} - 1\right) > \frac{x^2+2x}{7}\left[\frac{5}{x^2-x-6}\right] >$$
$$> \frac{x^2+2x}{7} \cdot \frac{5}{x^2-x-6}$$

(c) Moving on to the limit in these inequalities(the majoring and minoring method), we obtain the limit to the left:$l_s(-2) = \frac{2}{7}$ and the limit to the right: $l_r(-2) = \frac{2}{7}$, so $l = \frac{2}{7}$.

(B) If the double inequality cannot be used (3.2), for $\lim\limits_{x \to n} f(x)$, with $n \in \mathbb{Z}$, we calculate the lateral limits directly, keeping into consideration that:

- **if** $x \longrightarrow n$ **with** $x < n$, **then** $[x] = n - 1$.
- **if** $x \longrightarrow n$ **with** $x > n$, **then** $[x] = n$.

Example: For $\lim\limits_{n \to \infty}(-1^{[x]})/(x-1)$ we cannot use (3.2) because the expressions obtained for the calculation of the lateral limits don't have the same limit. That is why we use the fact that an the left of 1 we have $[x] = 0$, and on the right of 1 we have $[x] = 1$, so:

$$l_s(1) = \lim_{\substack{x \to 1 \\ x < 1}} \frac{(-1)^{[x]}}{x-1} = \lim_{\substack{x \to 1 \\ x < 1}} \frac{1}{x-1} = -\infty$$

$$l_d(1) = \lim_{\substack{x \to 1 \\ x > 1}} \frac{(-1)^{[x]}}{x-1} = \lim_{\substack{x \to 1 \\ x > 1}} \frac{-1}{x-1} = -\infty$$

It follows that $l = -\infty$

(C) If $x \to \infty$, we can replace $[x]$, keeping into consideration that for $x \in [m, m+1)$, we have: $[x] = m$ (obviously, $x \to \infty$ implies $m \to \infty$).

Example:

$$\lim_{x \to \infty} \left(1 + \frac{1}{[x]+(-1)^{[x]}}\right)^{[x]+(-1)^{[x]}}$$

The variable x that tends to infinity is certainly between two consecutive numbers: $m \le x \le m+1$, so:

$$\left(1 + \frac{1}{m+1}\right)^{m-1} \le \left(1 + \frac{1}{[x]+(-1)^{[x]}}\right)^{[x]+(-1)^{[x]}} <$$

$$\le \left(1 + \frac{1}{m+1}\right)^{m+1} \quad \textbf{and}$$

$$\lim_{m \to \infty} \left(1 + \frac{1}{m+1}\right)^{m} = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^{m+1} = e$$

Exercises

I. 1. $\lim\limits_{x \to 3} \dfrac{x^2 - 5x - 6}{4} \left[\dfrac{8}{x^2 - 8x + 15} \right]$

2. $\lim\limits_{x \to \infty} \dfrac{x^2 - 9x + 8}{x^2 + x + 1} \left[\dfrac{x^2 - 3x + 2}{x^2 - 12x + 32} \right]$

3. $\lim\limits_{x \to 0} \left[\dfrac{5}{\sin x} \right] \cdot \operatorname{tg} x$

4. $\lim\limits_{x \to 0} x \cdot \left[\dfrac{3\sin x + \cos x}{\sin x} \right]$

5. $\lim\limits_{x \to 0} \left[\dfrac{5x + 3}{\sin 2x} \right] \cdot \operatorname{tg} \dfrac{3x}{2}$

6. $\lim\limits_{x \to 0} x^2 \left[\dfrac{1}{x^2} \right]$

II. 1. $\lim\limits_{n \to \infty} \dfrac{[x] + [2^2 x] + \ldots + [n^2 x]}{n^3}$

2. $\lim\limits_{n \to \infty} \dfrac{[x] + [3^2 x] + [5^2 x] + \ldots + [(2n-1)^2 x]}{n^3}$

3. $\lim\limits_{n \to \infty} \dfrac{[x] + [2^k x] + \ldots + [n^k x]}{n^{k+1}}$

4. $\lim\limits_{n \to \infty} \dfrac{[x] \cdot 1! + [2x] \cdot 2! + \ldots + [nx] \cdot n!}{(n + 1)!}$

8. Using the definition of the derivative

As it is known, the derivative of a function is:

$$g'(x_0) = \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} \qquad (3.3)$$

if this limit exists and is finite. It can be used to (A) eliminate the indeterminations $\frac{0}{0}$ for limits of functions that can be written in the form:

$$\lim_{x \to x_o} \frac{g(x) - g(x_o)}{x - x_o},$$

x_o being a point in which g is differentiable.

Example:

$$\lim_{x \to a} \frac{a^x - a^a}{x - a}$$

We notice that by noting $g(x) = a^x$, we have $g(a) = a^a$ and the limit becomes:

$$\lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

This limit has the value $g'(a)$ because g is a function differentiable in a.

We have thus reduced the calculation of the given limit to the calculation of the value of $g'(a)$, calculation that can be made by deriving g:

$$g'(x) = a^x \ln a \text{, so } g'(a) = a^a \ln a = l.$$

Observation: this method can be used like this: "let $g(x) = a^x$. We have $g'(x) = a^x \ln a$. $g'(a) = a^a \ln a = l$. We don't recommend this synthetic method for the elaboration of the solution at exams because it can misguide the examiners. That is why the elaboration: "We observe that by noting $g(x) = \cdots$, we have $g(a) = \cdots$ and the limit becomes ... $= g'(a)$ because g is differentiable in a ..."is much more advisable.

(B) The calculation of the limits of sequences, using attached functions. The function attached to the sequence is not obtained, this time, by replacing n with x, in the expression of a_n, because for $x \to \infty$ we can't calculate the derivative, but by replacing n with $\frac{1}{x}$ (which leads to the calculation of the derivative in zero).

Example: For $\lim_{n \to \infty} n(\sqrt[n]{2} - 1)$ we consider the function $f(x) = \frac{2^x - 1}{x}$ obtained by replacing n with $\frac{1}{x}$ in the expression of

a_n. We calculate $\lim\limits_{n \to 0} f(x)$. For this we observe that by noting $g(x) = x^2$, we have $g(o) = 1$ and the limit becomes: $\lim\limits_{n \to 0} \frac{g(x)-g(o)}{x-0} = g'(0)$, because g is a differentiable function in zero. We have: $g'(x) = 2^x \ln 2$, so $g'(0) = \ln 2$. Then , according to the criterion with sequences, we obtain $\lim\limits_{n \to \infty} a_n = \ln 2$.

Exercises

I. 1. $\lim\limits_{x \to 1} \frac{\sqrt{x} - 1}{x - 1}$

2. $\lim\limits_{x \to 0} \frac{\sin x}{x}$

3. $\lim\limits_{x \to 1} \frac{\ln x}{x - 1}$

4. $\lim\limits_{x \to \frac{\pi}{4}} \frac{\sin x - \frac{\sqrt{2}}{2}}{x - \frac{\pi}{4}}$

5. $\lim\limits_{x \to 1} \frac{\operatorname{arctg} x - \frac{\pi}{4}}{x - 1}$

6. $\lim\limits_{x \to \frac{\pi}{4}} \frac{9^{\sin^2 x} - 3}{x - \frac{\pi}{4}}$

7. $\lim\limits_{x \to \pi} \frac{\cos x + 1}{x - \pi}$

8. $\lim\limits_{x \to \frac{\pi}{4}} \frac{\sin x \cdot \cos x - \frac{1}{2}}{x - \frac{\pi}{4}}$

II. For $a > 0$ calculate:

1. $\lim\limits_{x \to a} \dfrac{a^{x^x} - a^{a^a}}{x - a}$

2. $\lim\limits_{x \to a} \dfrac{a^{x^x} - a^{x^a}}{x - a}$

3. $\lim\limits_{x \to a} \dfrac{x^{tg\, x} - x^{tg\, a}}{x - a}$

4. $\lim\limits_{x \to a} \dfrac{x^{tg\, x} - x^{tg\, a}}{\sin bx - \sin ba}, \ a \neq \dfrac{k\pi}{2}$

5. $\lim\limits_{x \to 0} \left(\dfrac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}}$

6. $\lim\limits_{x \to 0} \dfrac{a^{x^2} - b^{x^2}}{(a^x - b^x)^2}$

7. $\lim\limits_{x \to 0} \dfrac{\log_b x - \log_a x}{x - a}$

ANSWER:

4. We divide the numerator and the denominator with $x - a$.

III. Show that we cannot use the definition of the derivative for the calculation of the limits:

1. $\lim\limits_{\substack{x \to 0 \\ x > 0}} \dfrac{\sqrt{x} - \cos x - 1}{x}$

2. $\lim\limits_{\substack{x \to 1 \\ x > 1}} \dfrac{1}{x^2 - 1} \cdot \ln \dfrac{2 \cdot \arcsin x}{\pi}$

IV. 1. $\lim\limits_{n \to \infty} n\left(\sqrt[n]{e} - 1 \right)$

2. $$\lim_{n \to \infty} \frac{\sqrt[n]{2} - 1}{\sqrt[n]{3} - 1}$$

3. $$\lim_{n \to \infty} n\left(\sqrt[n]{a_1} + \sqrt[n]{a_2} - 2 \right)$$

4. $$\lim_{n \to \infty} n\left(\sqrt[n]{a_1} + \sqrt[n]{a_2} + \ldots + \sqrt[n]{a_k} - k \right)$$

5. $$\lim_{n \to \infty} \left(\frac{\sqrt[n]{a_1} - \sqrt[n]{a_2}}{2} \right)^n, \text{with } a_1, a_2 > 0$$

6. $$\lim_{n \to \infty} \left(\frac{\sqrt[n]{a_1} + \sqrt[n]{a_2} + \ldots + \sqrt[n]{a_k}}{k} \right)^n,$$

with $a_1, a_2, \ldots, a_k > 0$

7. $$\lim_{n \to \infty} \frac{n\left(\sqrt[n]{n} - 1 \right)}{\ln n}$$

ANSWERS:

2. We consider the function $f(x) = \frac{2^x - 1}{3^x - 1}$, obtained by replacing the sequence a_n on n with $\frac{1}{x}$. We calculate $\lim_{x \to 0} f(x)$. To use the definition of the derivative, we divide the numerator and the denominator with x, so:

$$f(x) = \frac{\frac{2^x - 1}{x}}{\frac{3^x - 1}{x}}$$

We observe that by noting $g(x) = 2^x$, we have $g(0) = 1$ and the limit from the numerator becomes: $\lim_{x \to 0} \frac{g(x) - g(0)}{x} = g'(x)$ because g is differentiable in zero.

$g'(x) = 2^x \ln 2$, so, according to the criterion with sequences $l_1 = g'(0) = \ln 2$. Analogously, by noting $h(x) = 3^x$, the limit from the numerator becomes: $\lim_{x \to 0} \frac{h(x)-h(0)}{x} = h'(0)$, because h is differentiable in zero.

$h(x) = 3^x \ln 3$, so, according to the criterion with sequences$l_2 = h'(0) = \ln 3$, and $l = \frac{\ln 2}{\ln 3}$.

5. The limit contains an indetermination with the form 1^∞, so we will first apply method 4.

$$\lim_{n \to \infty} \left[\frac{\sqrt[n]{a_1} + \sqrt[n]{a_2}}{2} \right]^n =$$

$$= \lim_{n \to \infty} \left[1 + \frac{\sqrt[n]{a_1} + \sqrt[n]{a_2}}{2} - 1 \right]^n =$$

$$= \lim_{n \to \infty} \left[\left[1 + \frac{\sqrt[n]{a_1} + \sqrt[n]{a_2} - 2}{2} \right]^{\frac{2}{\sqrt[n]{a_1} + \sqrt[n]{a_2} - 2}} \right]^{n \cdot \frac{\sqrt[n]{a_1} + \sqrt[n]{a_2} - 2}{2}} = e^{\lim_{n \to \infty} n \frac{\sqrt[n]{a_1} + \sqrt[n]{a_2} - 2}{2}}$$

We now consider the function $f(x) = \frac{1}{x}\,(a_1^x + a_2^x - 2)$ obtained by replacing n with $\frac{1}{x}$ in the expression above. We calculate $\lim_{x \to 0} f(x)$. For this, we observe that by noting $g(x) = a_1^x + a_2^x$ we have $g(0) = 2$ and the limit can be written: $\lim_{x \to 0} \frac{g(x)-g(0)}{x} = g'(0)$ because g is differentiable in zero.

Now $g'(x) = a_1^x \ln a_1 + a_2^x \ln a_2$, so $g'(0) = \ln a_1 + \ln a_2$

According to the criterion with sequences:

$$\lim_{n \to \infty} n\left(\sqrt[n]{a_1} + \sqrt[n]{a_2} - 2\right) = \ln(a_1 a_2),$$

and the initial limit is $\sqrt{a_1 . a_2}$.

9. Using L'Hospital's theorem

The theorem. [l'Hospital (1661-1704)] If the functions f and g fulfill the conditions:

1. are continuous on $[a, b]$ and differentiable on $(a, b)\{x_o\}$
2. $f(x_o) = g(x_o) = 0$
3. g' is not annulled in a vicinity of x_o
4. there exists $\lim_{x \to x_o} \frac{f'(x)}{g'(x)} = \alpha$, finite or infinite,

then there exists $\lim_{x \to x_o} \frac{f(x)}{g(x)} = \alpha$.

This theorem can be used for:

(A) The calculation of limits such as $\lim_{x \to x_o} \frac{f(x)}{g(x)}$ when these contain an indetermination of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

(B) The calculation of limits such as $\lim_{x \to x_o} f(x).g(x)$ when these contain an indetermination of the form $\infty.0$.

This indetermination can be brought to form (A) like this:

$$\lim_{x \to x_o} f(x)\cdot g(x) = \lim_{x \to x_o} \frac{f(x)}{\frac{1}{g(x)}}$$

(indetermination with the form $\frac{\infty}{\infty}$)

$$\lim_{x \to x_o} f(x)\cdot g(x) = \lim_{x \to x_o} \frac{g(x)}{\frac{1}{f(x)}}$$

(indetermination with the form $\frac{0}{0}$)

Observation: sometimes it is essential if we write the indetermination $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example:

$$\lim_{x \to 0} \frac{1}{x^2} \cdot e^{\frac{-1}{x^2}}$$

If we write:

$$\lim_{x \to 0} \frac{1}{x^2} \cdot e^{\frac{-1}{x^2}} = \lim_{x \to 0} \frac{e^{\frac{-1}{x^2}}}{x^2}$$

(indetermination $\frac{0}{0}$) and we derive, we obtain:

$$l = \lim_{x \to 0} \frac{\frac{2}{x^3} \cdot e^{\frac{-1}{x^2}}}{2x} = \lim_{x \to 0} \frac{e^{\frac{-1}{x^2}}}{x^4}$$

(indetermination with the form $\frac{0}{0}$ with a higher numerator degree).

If we highlight the indetermination $\frac{\infty}{\infty}$:

$$\lim_{x \to 0} \frac{1}{x^2} \cdot e^{\frac{-1}{x^2}} = \lim_{x \to 0} \frac{\frac{1}{x^2}}{e^{\frac{1}{x^2}}}$$

and we derive, we obtain $l = 0$.

(C) The calculation of limits such as $\lim\limits_{x \to x_o} (f(x) - g(x))$, with indetermination $\infty - \infty$.

This indetermination can be brought to form (B) giving $f(x)$ or $g(x)$ as forced common factor.

We have, for example:

$$\lim_{x \to x_0} \left(f(x) - g(x)\right) = \lim_{x \to x_0} f(x) \cdot \left(1 - \frac{g(x)}{f(x)}\right)$$

and because

$$f(x) \xrightarrow{x \to x_0} \infty$$

we distinguish two cases:

– if $\lim_{x \to x_0} \left(1 - \frac{g(x)}{f(x)} \right) = a \neq 0$

(namely, $\frac{g(x)}{f(x)}$ doesn't tend to 1), we have:

$$\lim_{x \to x_0} \left(f(x) - g(x) \right) = \infty \cdot a = \infty \cdot \operatorname{sgn} a = \pm \infty .$$

– if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$,

the indetermination is of the form (B).

(D) The calculation of limits such as $\lim_{x \to x_0} f(x)^{g(x)}$, with indeterminations $1^\infty, 0^0, \infty^0$.

These indeterminations can be brought to the form (C), using the formula:

$$A = a^{\log_a A}$$

that for $a = e$, for example, becomes $A = e^{\ln A}$, so:

$$\lim_{x \to x_0} f(x)^{g(x)} = \lim_{x \to x_0} e^{\ln f(x)^{g(x)}} =$$

$$= e^{\lim_{x \to x_0} g(x) \cdot \ln f(x)}$$

The last limit contains an indetermination of the form (C).

For sequences, this method is applied only through one of the two functions attached to the sequence (obtained by replacing n with x or by replacing n with $\frac{1}{x}$ in the expression of a_n). According

to the criterion with sequences it follows that the limit of the sequence is equal to the limit of the attached function.

Attention: by replacing n with x we calculate the limit in ∞, and by replacing n with $\frac{1}{x}$, we calculate the limit in 0.

Exercises

I. Calculate:

1. $\lim\limits_{x \to 0} \frac{e^{ax} - e^{-2ax}}{\ln(1 + x)}$

2. $\lim\limits_{x \to \infty} \frac{\log_a x}{x^k}$, if $a > 1$

(a polynomial increases faster to infinity than a logarithm)

3. $\lim\limits_{x \to \infty} \frac{a^x}{x^k} = \infty$, if $a > 1$

(a polynomial increases slower to infinity than an exponential)

4. $\lim\limits_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x - 1} \right)$

5. $\lim\limits_{x \to 0} \ln(1 + \sin^2 x) \cdot \operatorname{ctg} \ln^2(1 + x)$

6. $\lim\limits_{x \to 0} \frac{x - \operatorname{tg} x}{x - \sin x}$

7. $\lim\limits_{x \to 1} \frac{a^{\ln x} - x}{\ln x}$

8. $\lim\limits_{x \to \frac{\pi}{2}} (\sin x)^{\operatorname{tg} x}$

9. $\lim\limits_{n \to \infty} n \cdot (\sqrt[n]{e} - 1)$

10. $\lim\limits_{n \to \infty} n \cdot \sin \frac{1}{n}$

11. $\lim\limits_{n \to \infty} \frac{n}{e^n}$

12. $\lim\limits_{n \to \infty} \left(n + e^n \right)^{\frac{1}{n}}$

II. Show that l'Hospital's rule cannot be applied for:

1. $\lim\limits_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x}$ 2. $\lim\limits_{x \to \infty} \frac{x - \sin x}{x + \cos x}$

ANSWER:

1. For $f(x) = x^2 sin\frac{1}{x}$ and $g(x) = \sin x$ we have:

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{2x \cdot \sin \frac{1}{x} - \cos \frac{1}{x}}{\cos x}$$

- doesn't exist, so the condition 4 from the theorem isn't met. Still, the limit can be calculated observing that:

$$\lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x} = \lim_{x \to 0} \frac{x}{\sin x} \cdot x \cdot \sin \frac{1}{x} =$$

$$= 1 \cdot 0 = 0.$$

10. Using the criterion with sequences (the Heine Criterion)

The criterion enunciation:

$$\lim_{x \to x_0} f(x) = l \iff \Big[\forall \, (x_n)_{n \in N}$$

with the properties:

$$\left.\begin{array}{l} a)\ x_n \longrightarrow x_0 \\ b)\ x_n \in D \\ c)\ x_n \neq x_0 \end{array}\right\} \Rightarrow \lim_{n \to \infty} f(x_n) = l \Big]$$

Observation: If $x_0 = \infty$, condition $c)$ doesn't make sense anymore.

This criterion cannot be used to eliminate indeterminations because the sequence $(x_n)_{n\in\mathbb{N}}$ from the enunciation is random, so, by modifying the sequences (an infinite number of sequences), we cannot eliminate the indetermination. Still, the criterion can be used at:

(A) The calculation of limits that don't contain indeterminations.

Example: Using the criterion with sequences, show that:

$$\lim_{x \to 2} \frac{3x+5}{2x+3} = \frac{11}{7}$$

Solution: (a) we have to show that:

$\forall\ (x_n)_{n\in\mathbb{N}}$, with the properties:

$$\begin{aligned}
&a)\ x_n \longrightarrow 2 \\
&b)\ x_n \in \mathbb{R}\setminus\left\{ \frac{3}{2} \right\} \Rightarrow \lim_{n \to \infty} f(x_n) = \frac{11}{7} \\
&c)\ x_n \neq 2
\end{aligned}$$

(b) let $(x_n)_{n\in\mathbb{N}}$ be any sequence with the properties $a), b), c)$

(c) in $\lim\limits_{n \to \infty} f(x_n)$ we use the properties of the operations of limits of sequences to highlight $\lim\limits_{n \to \infty} x_n$ (it is possible because there are no indeterminations), then we use the fact that $\lim\limits_{n \to \infty} x_n = x_0$:

$$\lim_{n \to \infty} \frac{3\cdot x_n + 5}{2\cdot x_n + 3} = \frac{\lim\limits_{n \to \infty} (3x_n + 5)}{\lim\limits_{n \to \infty} (2x_n + 3)} =$$

$$= \frac{3 \cdot \lim_{n \to \infty} x_n + 5}{2 \cdot \lim_{n \to \infty} x_n + 3} = \frac{3 \cdot 2 + 5}{2 \cdot 2 + 3} = \frac{11}{7}.$$

(B) To show that a function doesn't have a limit in a point.

To accomplish this we can proceed as follows:

1_B. we find two sequences: $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ with the properties $a), b), c)$ so that:

$$\lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(y_n)$$

or:

2_B. we find a single sequence with the properties $a), b), c)$ so that $\lim_{n \to \infty} f(x_n)$ doesn't exist.

Example: $\lim_{x \to \infty} \sin x$ doesn't exist

1_B. let $x_n = 2n\pi$ and $y_n = 2n\pi + \frac{\pi}{2}$. We have:

a) $x_n \longrightarrow \infty$, $y_n \longrightarrow \infty$

b) $x_n, y_n \in \mathbb{R}$

and $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} \sin 2n\pi = 0$,

$\lim_{n \to \infty} f(y_n) = \lim_{n \to \infty} \sin(2n\pi + \frac{\pi}{2}) = 1.$

So the limit doesn't exist.

2_B. let $x_n = \frac{n\pi}{2}$. We have:

a) $x_n \longrightarrow \infty$

b) $x_n \in \mathbb{R}$

and $\lim_{n \to \infty} \sin \frac{n\pi}{2}$ doesn't exist.

(C) The criterion with sequences can be used in the calculation of the limits of sequences as follows:

(a) let $f(x)$ be the function attached to the sequence, by the replacement of, for example, n with x in the expression of a_n (or of n with $\frac{1}{x}$, but then we calculate $\lim\limits_{x \to 0} f(x)$)

(b) we calculate $\lim\limits_{x \to \infty} f(x) = l$ ($\lim\limits_{n \to 0} f(x)$, respectively).

(c) according to the criterion with sequences $\lim f(x) = l$ means: for any sequence $(x_n)_{n \in \mathbb{N}}$ with the properties:

a) $x_n \longrightarrow \infty$

b) $x_n \in \mathbb{D}$

we have $\lim\limits_{n \to \infty} f(x_n) = l$

We observe that the sequence $x_n = n$ fulfills the conditions $a)$ and $b)$ and, moreover, for this sequence we have:

$$f(x_n) = a_n, \text{ so } \lim_{n \to \infty} a_n = \lim_{n \to \infty} f(n) = l.$$

Exercises

I. Using the criterion with sequences, calculate:

1. $\lim\limits_{x \to 1} \dfrac{4x + 5}{2x^2 - 3}$

2. $\lim\limits_{x \to 1} \left(\sqrt{2x + 2} - \operatorname{arctg} x \right)$

3. $\lim\limits_{x \to \frac{\pi}{2}} \left(\operatorname{ctg} x + x \right)$

4. $\lim\limits_{x \to 0} \sqrt[3]{\cos x}$

5. $\lim\limits_{x \to 3} \sqrt{3^x + 9}$

II. Show that the following don't exist:

1. $\lim_{x \to \infty} \cos x$

2. $\lim_{x \to \infty} tg\, x$

3. $\lim_{x \to \infty} e^{\sin x}$

4. $\lim_{x \to \infty} \sqrt{1 + \sin^2 x}$

5. $\lim_{x \to \infty} \frac{2 \cdot \sin x + \cos x}{3 + \cos 2x}$

6. $\lim_{x \to \infty} (x - [x])$

7. If f has a limit to its left (right) in x_0, then:

$$l_s(x_o) = \lim_{n \to \infty} f(x_o - \frac{1}{n})$$

$(l_d(x_o) = \lim_{n \to \infty} f(x_o + \frac{1}{n}$ **respectively).**

(XI Grade Handbook)

8. If $\lim_{n \to \infty} f(x) = l$, then $\lim_{n \to \infty} f(n) = l$, but not the other way around.

9. $\lim_{x \to \infty} f(x)$ doesn't exist if $f: D \to \mathbb{R}$ is a periodic, non-constant function. Consequences: the following do not exist:

$$\lim_{x \to \infty} \sin x \,, \lim_{n \to \infty} \cos x \,, \lim_{n \to \infty} tg\, x \,, \lim_{x \to \infty} ctg\, x \,.$$

Answer: 9. This exercise is the generalization of exercises 1.-6. For the solution, we explain the hypothesis:

– f – periodic $\Leftrightarrow \exists\, T > 0 \;\; \forall x \in D,\ f(x + T) = f(x)$

– f – nonconstant $\Leftrightarrow \exists\, a, b \in D \;\; f(a) \neq f(b)$

We now use method 1_B. The following sequences fulfill conditions $a)$ and $b)$:

$$x_n = a + nT \,, \quad y_n = b + nT$$

and:

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(a+nT) = \lim_{n\to\infty} f(a) = f(a)$$

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(b+nT) = \lim_{n\to\infty} f(b) = f(b)$$

III. Determine the limit points of the functions: $f, g, f \circ g, g \circ f$, for:

$$1.\quad f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 2 & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

$$g(x) = \begin{cases} 2x+3 & x \in \mathbb{Q} \\ 3x+1 & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

$$2.\quad f(x) = \begin{cases} \sin x & x \in \mathbb{Q} \\ \cos x & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

$$g(x) = \begin{cases} \frac{\pi}{2} & x \in \mathbb{Q} \\ -\frac{\pi}{2} & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

$$3.\quad f(x) = \begin{cases} \sqrt[3]{x} & x \in \mathbb{Q} \\ 8 & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

Answer: let $x_0 \in D = \mathbb{R}$ randonm. We check if f has a limit in x_0. For this we observe it is essential if $(x_n)_{n\in\mathbb{N}}$ is rational or irrational. We firstly consider the situation where the sequence $(x_n)_{n\in\mathbb{N}}$ is made up only of rational points (or only of irrational points).

Let $(x_n)_{n\in\mathbb{N}}$ be **a sequence of rational points** with the properties $a), b), c)$. We have:

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} 2 = 2.$$

So the function doesn't have a limit in any point x_0 for which $x_0^2 \neq 2$, i.e. x_0 and ± 2. For $x_0 = \sqrt{2}$ we consider a random sequence $(x_n)_{n\in\mathbb{N}}$ with the properties $a), b), c)$. $E1$decomposes in two subsequences: $(x'_n)_{n\in\mathbb{N}}$ made up of solely rational terms and $(x'_n)_{n\in\mathbb{N}}$ made up of irrational terms. Because:

$$\lim_{n\to\infty} f(x'_n) = \lim_{n\to\infty} (x'_n)^2 = (\sqrt{2})^2 = 2$$

$$\text{and } \lim_{n\to\infty} f(x''_n) = \lim_{n\to\infty} 2 = 2,$$

we deduce that:

$$\lim_{x\to\sqrt{2}} f(x) = 2$$

The same goes for $x = -\sqrt{2}$.

Specific methods for sequences

11. Any bounded and monotonic sequence is convergent

Applying this method comes back to the study of monotony and bounding, and for the determination of the limit we pass over to the limit in the recurrence relation of the given sequence. If such a relation is not initially provided, it can be obtained at the study of the monotony.

Example: $a_n = \frac{n!}{n^n}$

(a) **For the study of monotony** and bounding we cannot apply the method of the attached function ($f(x) = \frac{x!}{x^x}$ doesn't make sense). We employ the method of the report (we begin by studying the monotony because (1) if the sequence is monotonic, at least half of the bounding problem is solved, as it has been already proven; (2) if the sequence isn't monotonic, we can't apply the method so we won't study the bounding.

We therefore have:

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)!}{(n+1)^{n+1}}}{\dfrac{n!}{n^n}} = \frac{n^n}{(n+1)^n} =$$

$$= \left(\frac{n}{n+1}\right)^n \xrightarrow[n\to\infty]{} \frac{1}{e} < 1$$

So the sequence is decreasing.

(b) **The bounding**. The sequence is bounded between 0 and a_1, being decreasing and with positive terms. Therefore $\lim\limits_{n\to\infty} a_n$ exists.

(c) For the calculation of l we observe that studying the monotony we have obtained the recurrence relation:

$$\frac{a_{n+1}}{a_n} = \left(\frac{n}{n+1}\right)^n \Leftrightarrow a_{n+1} = a_n\left(\frac{n}{n+1}\right)^n$$

Moving on to the limit in this equality, we obtain:

$$l = l\cdot\frac{1}{e}. \Leftrightarrow l = 0.$$

EXERCISES:

1. $a_n = \dfrac{2^n}{(n!)^2}$

2. $a_n = \dfrac{1\cdot 3\cdot 5\cdot \ldots \cdot(2n-1)}{3^n\cdot n!}$

3. $a_n = \sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}$ – **n radicals**

4. $a_n = \left(\dfrac{2}{2!} + \dfrac{7}{3!} + \ldots + \dfrac{n^2-2}{n!}\right) + \dfrac{n+2}{n!}, \quad n \geq 2$

5. $a_n = \left(\dfrac{7}{1\cdot 8} + \dfrac{7}{8\cdot 15} + \ldots +\right.$

$$+ \frac{7}{(7n-6)\cdot(7n+1)} \Big] + \frac{1}{7n+1}$$

6. $$a_n = \sum_{k=1}^{n} \frac{2k+3}{5^k}$$

7. $$a_n = \frac{1}{b_1} + \frac{1}{b_2} + \dots + \frac{1}{b_n},$$

where $(b_n)_{n\in\mathbb{N}}$ fulfills conditions:

a) $\lim_{n\to\infty} b_n = \infty$, b) $b_n(b_{n+1} - b_n) \leq 1$.

8. $$a_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}$$

9. $$a_n = \frac{1}{\sqrt[k]{1}} + \frac{1}{\sqrt[k]{2}} + \dots + \frac{1}{\sqrt[k]{n}}$$

ANSWERS:

3. We observe the recurrence relation:

$$a_{n+1} = \sqrt{2 + a_n}.$$

a) **The monotony**:

$$a_{n+1} - a_n = \sqrt{2 + a_n} - a_n.$$

In order to compare this difference to zero, we make a choice, for example $a_{n+1} \geq a_n$, that we then transform through equivalences:

$$a_{n+1} \geq a_n \Leftrightarrow \sqrt{2 + a_n} \geq a_n \Leftrightarrow 2 + a_n \geq a_n^2 \Leftrightarrow$$

$$a_n^2 - a_n - 2 \leq 0; \quad x^2 - x - 2 = 0 \Rightarrow$$

$$x_1 = -1, \quad x_2 = 2$$

So $a_{n+1} \geq a_n \Leftrightarrow a_n \in (-1,2)$.

We have thus reached the conclusion that **the sequence is increasing if and only if it is bounded between -1 and 2**. We obviously have $a_n > 0 > -1$ so we still have to prove that $a_n < 2$. This inequality can be proven by induction:

$$P(n): \quad a_n \le 2$$

Verification:

$$P(1): \quad a_1 \le 2 \quad \Leftrightarrow \quad \sqrt{2} \le 2 \text{ (true).}$$

$$P(n) \quad \Rightarrow \quad P(n+1)$$

$$P(n+1): \ a_{n+1} \le 2 \Leftrightarrow \sqrt{2 + a_n} \le 2 \Leftrightarrow$$

$$\Leftrightarrow 2 + a_n \le 4 \Leftrightarrow a_n \le 2.$$

So we have proven the monotony and the bounding. Therefore $l = \lim_{n \to \infty} a_n$ exists. To determine l we move on to the limit in the recurrence relation (this time this type of relation is given initially):

$$a_{n+1} = \sqrt{2 + a_n} \Rightarrow \lim_{n \to \infty} a_{n+1} =$$

$$= \lim_{n \to \infty} \sqrt{2 + a_n} \Rightarrow l = \sqrt{2 + l}$$

$$\Rightarrow l^2 = 2 + l \Rightarrow l = -1 \quad \text{or} \quad l = 2.$$

As $l = -1$ is impossible, because $a_n > 0$, we deduce $l = 2$.

6. We decompose the fraction $\frac{2k+3}{5^k}$ in a difference of consecutive terms:

$$\frac{2k+3}{5^k} = \frac{A \cdot k + B}{5^k} + \frac{C \cdot k + D}{5^k} \qquad (3.4)$$

To determine the four parameters:

a) we use the identification method and

b) we state the condition that the terms from the right member in (3.4) be consecutive.

$$a) \quad 2k + 3 \equiv 5Ak + 5B - Ck - D \quad \Leftrightarrow$$

$$\Leftrightarrow \quad 2k + 3 = k(5A - C) + 5B - D$$

c) the denominators are consecutive, so we state the condition that the numerators be in the same relation as consecutives; the denominator of the first fraction is obtained from the denominator of the second one, by replacing k with $k-1$, so we must have the same relation between the numerators:

$$A \cdot k + B = C(k-1) + D$$

$$\Leftrightarrow \qquad A \cdot k + B = C \cdot k + (D - C)$$

We have thus obtained the system:

$$\begin{cases} 5A - C = 2 \\ 5B - D = 3 \\ A = C \\ B = D - C \end{cases}$$

with the solution:

$$A = C = \frac{1}{2}, \quad B = \frac{-3}{8}, \quad D = \frac{1}{8}$$

It follows that:

$$a_n = \sum_{k=1}^{n} \frac{2k+3}{5^k} =$$

$$= \sum_{k=1}^{n} \left(\frac{\frac{1}{2}k - \frac{3}{8}}{5^{k-1}} - \frac{\frac{1}{2}k + \frac{1}{8}}{5^k} \right) =$$

$$= \sum_{k=1}^{n} \frac{\frac{1}{2}k - \frac{3}{8}}{5^{k-1}} - \sum_{k=1}^{n} \frac{\frac{1}{2}k + \frac{1}{8}}{5^k} =$$

$$= \left[\frac{1}{8} + \frac{1}{5 \cdot 8} + \frac{1}{5^2 \cdot 8} + \ldots + \frac{\frac{1}{2}n - \frac{3}{8}}{5^{n-1}} \right] -$$

$$- \left[\frac{5}{5 \cdot 8} + \frac{9}{5^2 \cdot 8} + \ldots + \frac{\frac{1}{2}n + \frac{1}{8}}{5^n} \right] =$$

$$= \frac{1}{8} - \frac{\frac{1}{2}n + \frac{1}{8}}{5^n},$$

consequently:

$$\lim_{n \to \infty} a_n = \frac{1}{8}.$$

7. We have:

$$\frac{1}{b_n} \geq \frac{1}{b_{n+1}} - \frac{1}{b_n},$$

so:

$$a_n \geq (b_2 - b_1) + (b_3 - b_2) + \ldots +$$
$$+ (b_{n+1} - b_n) = b_{n+1} - b_1 \longrightarrow \infty.$$

8. We apply the conclusion from 7.

II. Study the convergence of the sequences defined through:

1. $a_{n+1} = \dfrac{5 \cdot a_n + 3}{a_{n+1}}$, with a_0 given.

2. $x_{n+1} = x_n^2 - 2 \cdot x_n + 2$, with x_1 given.

3. $x_{n+1} = \dfrac{x_n}{1 + n \cdot x_n^2}$, with $x_1 > 0$ given.

4. $a_n = \dfrac{1}{2} \cdot \left(a_{n-1} + \dfrac{\alpha}{a_{n-1}} \right)$, with $a_0 > 0$ given.

5. $a_n = \dfrac{x}{2} - \dfrac{a_{n-1}}{2}$, with $x \in [0, 1]$

6. $x_n = \dfrac{2 \cdot a \cdot x_{n-1}}{a + x_{n-1}}$, with $a, x_0 > 0$ given.

7. $a_n = \int_n^{n+1} e^{-x^2} dx$.

Indications:

4. sequence with positive terms (induction) and decreasing.

5. $a_{2k+2} - a_{2k} > 0$, $a_{2k+3} - a_{2k+1} < 0$

and $|a_k| < \frac{1}{2}$.

Let l_1 and l_2 be the limits of the subsequences of even and uneven value. Show that $l_1 = l_2$.

12. Using the Cesaro-Stolz and Rizzoli Lemmas

Lemma: (Cesaro-Stolz) Let $(\alpha_n)_{n\in\mathbb{N}}$ be a random sequence and $(\beta_n)_{n\in\mathbb{N}}$ a strictly increasing sequence, having the limit infinite. If $\lim\limits_{n \to \infty} \frac{\alpha_{n+1}-\alpha_n}{\beta_{n+1}-\beta_n} = l$ exists – finite or infinite, then:

$$\lim_{n \to \infty} \frac{\alpha_n}{\beta_n} = l.$$

This lemma can be used to calculate the limits of sequences that can be expressed as a fraction:

$$\lim_{n \to \infty} \frac{\alpha_n}{\beta_n} \tag{3.5}$$

and of the limits of the form:

$$\lim_{n \to \infty} \frac{\alpha_{n+1} - \alpha_n}{\beta_{n+1} - \beta_n}$$

presupposing that this limit exists and $(\beta_n)_{n\in\mathbb{N}}$ is strictly monotonic.

CONSEQUENCES:

(A) If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, then $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = l$.

Indeed,

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} e^{\ln \sqrt[n]{a_n}} = e^{\lim\limits_{n\to\infty} \frac{\ln a_n}{n}} =$$

$$= e^{\lim\limits_{n\to\infty} \frac{\ln a_{n+1} - \ln a_n}{n+1-n}} = e^{\lim\limits_{n\to\infty} \ln \frac{a_{n+1}}{a_n}} =$$

$$= e^{\ln \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}} = e^{\ln l} = l .$$

Example:

$$\lim_{n\to\infty} \sqrt[n]{n} = 1 ,$$

because for $a_n = n$, we have:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

(B) The limits of the arithmetic, geometric and harmonic averages of the first n terms of a sequence having the limit l, have the value l also:

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \ldots + a_n}{n} = \lim_{n\to\infty} \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} =$$

$$= \lim_{n\to\infty} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}} = \lim_{n\to\infty} a_n$$

Observation: a variant for the Cesaro-Stolz lemma for the case when $a_n, \beta_n \to 0$ has been proven by I.Rizzoli [*Mathematic Gazette*, no 10-11-12, 1992, p. 281-284]:

Lemma: (I.Rizzoli) If $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ are two sequences of natural numbers that fulfill the conditions:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$; $\lim\limits_{n\to\infty} \beta_n = 0$

(ii) the sequence $(\beta_n)_{n\in\mathbb{N}}$ is strictly monotonic (increasing or decreasing)

(iii) there exists:

$$l = \lim_{n\to\infty} \frac{\alpha_{n+1} - \alpha_n}{\beta_{n+1} - \beta_n}$$

Then: $\lim\limits_{n\to\infty} \frac{\alpha_n}{\beta_n} = l.$

EXERCISES:

I 1. $a_n = \dfrac{\ln n!}{n^2}$

2. $a_n = \dfrac{1^2 + 2^2 + \ldots + n^2}{n^3}$

3. $a_n = \dfrac{1^p + 2^p + \ldots + n^p}{n^{p+1}}$

4. $a_n = \dfrac{1^p + 3^p + \ldots + (2n-1)^p}{n^{p+1}}$

5. $a_n = \dfrac{1^p + 2^p + \ldots + n^p}{n^p} - \dfrac{n}{p}$

6. $a_n = \dfrac{\sum\limits_{k=1}^{n} (4k-1)^p}{n^{p+1}}$

7. $a_n = \dfrac{1\cdot 3 + 3\cdot 5 + \ldots + (2n-1)(2n+1)}{n^2}$

8. $a_n = \dfrac{1\cdot 2\cdot \ldots\cdot k + 2\cdot 3\cdot\ldots\cdot(k+1) + \cdots + (n-k+1)\cdot(n-k+2)\cdot\ldots\cdot n}{n^k}$

9. $a_n = \dfrac{1 + \sqrt[4]{2!} + \sqrt[9]{3!} + \ldots + \sqrt[n^2]{n!}}{n}$

II. 1. $a_n = \dfrac{\sqrt[n]{n!}}{n}$

2. $a_n = \sqrt[n]{\dfrac{3^{3n}(n!)^3}{(3n)!}}$

3. $a_n = \sqrt[n]{\dfrac{(n+1)(n+2)\ \ldots\ (2n)}{n!}}$

4. $a_n = \sum_{k=1}^{n} \left(\sqrt{1 + \dfrac{k^m}{n^{m+1}}} - 1 \right)$

(*Indications*: amplification with the conjugate, the minoring/majoring method, the Cesaro-Stolz lemma).

5. Let $x_{n+1} = \ln(1 + x_n)$, $x_0 > 0$. Using the Cesaro-Stolz lemma show that $\lim\limits_{n \to \infty} n.x_n = 2$.

(Indications: $nx_n = \frac{n}{\frac{1}{x_n}}$)

III. Calculate the limits of the following sequences, presupposing they exist:

1. $L_n = \sqrt[n+1]{(n + 1)!} + \sqrt[n]{n!}$ **(Lalescu sequence)**

2. $a_n = \sqrt[n+1]{\dfrac{(n + 2)(n + 3)\ \ldots\ (2n + 2)}{(n + 1)!}} -$

$- \sqrt[n]{\dfrac{(n +1)(n + 2)\ \ldots\ (2n)}{n!}}$

3. $a_n = n^p \left[\dfrac{\pi^2}{6} - \left(1 + \dfrac{1}{2^2} + \ldots. + \dfrac{1}{n^2} \right) \right]$

knowing that:

$$\lim_{n\to\infty}\left[1+\frac{1}{2^2}+\dots+\frac{1}{n^2}\right]=\frac{\pi^2}{6}$$

4. $a_n = n^p\left(\frac{\sin 1}{n^2+1}+\frac{\sin 2}{n^2+2}+\dots+\frac{\sin n}{n^2+n}\right)$

5. $a_n = 1^p + 2^p + \dots n^p - \frac{n^{p+1}}{p} - \frac{n^p}{2}$

6. $a_n = n^p\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n - c\right)$

where $p > 0$ and c is Euler's constant.

Answers:

1. We apply the Cesaro-Stolz lemma for:

$\alpha_n = \sqrt{(n+1)!}$ and $\beta_n = n$.

We have:

$$\lim_{n\to\infty} \alpha_n / \beta_n = \lim_{n\to\infty} \sqrt{n!/n^n} = 1/e$$

and through the hypothesis it follows that:

$\lim (\alpha_{n+1} - \alpha_n)/(\beta_{n+1} - \beta_n) = l$, so $l = 1/e$.

6. We consider:

$\alpha_n = 1 + \frac{1}{2} + \dots \frac{1}{n} - \ln n - c$ and $\beta_n = \frac{1}{n^p}$.

We have:

$$\frac{\alpha_{n+1}-\alpha_n}{\beta_{n+1}-\beta_n} = \frac{\frac{1}{n+1} - \ln\frac{n}{n+1}}{\frac{1}{(n+1)^p} - \frac{1}{n^p}} =$$

$$= \frac{n^p(n+1)^p\left(\frac{1}{n+1}+\ln\frac{n}{n+1}\right)}{n^p - (n+1)^p} =$$

$$= \frac{n^{2p}\left(1+\frac{1}{n}\right)^p\left(\frac{1}{n+1}+\ln\frac{n}{n+1}\right)}{-n^{p-1}\left(C_p^1 + \frac{1}{n}C_p^2 + \dots\right)}$$

and keeping into account that:

$$\lim_{n \to \infty} \frac{\frac{1}{n+1} + \ln\frac{n}{n+1}}{\frac{1}{n^2}} = -\frac{1}{2}$$

(we can use the attached function), we obtain:

$$l = \begin{cases} 0 & \text{if} \quad p \in (0, 1) \\ \frac{1}{2} & \text{if} \quad p = 1 \\ \infty & \text{if} \quad p > 1 \end{cases}$$

13. Utilization of Lagrange's theorem

Theorem: (Lagrange, (1736-1813)). If $f: [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b),

then: $\exists c \in (a, b) a.i. \frac{f(b)-f(a)}{b-a} = f'(c)$.

This theorem can be used for:

(A) The calculation of the limits of sequences in which we can highlight an expression with the form:

$$\frac{f(b) - f(a)}{b - a} \tag{3.6}$$

for the attached function.

Example:

$$a_n = \sqrt{n}\left[\left(1 + \frac{1}{n+1}\right)^{n+1} - \left(1 + \frac{1}{n}\right)^n\right]$$

Solution:

a) we highlight an expression of the form (3.6), considering:

$$f(x) = \left(1 + \frac{1}{x}\right)^x, \ f : [n, n+1] \longrightarrow \mathbb{R}, \ n \in \mathbb{N}^*.$$

b) we apply Lagrange's theorem on function f on the interval $[n, n+1]$:

$$\exists\ c \in (n, n+1)\quad \frac{f(n+1) - f(n)}{n+1-n} = f'(c_n).$$

c) we replace in a_n the expression $\frac{f(b)-f(a)}{b-a}$ with $f'(c_n)$. We obtain:

$$a_n = \sqrt{n}\left(1 + \frac{1}{c_n}\right)^{c_n} \cdot \left[\ln\left(1 + \frac{1}{c_n}\right) - \frac{1}{1 + c_n}\right]$$

d) we use the inequalities: $n < c_n < n+1$ to obtain convenient majorings (minorings). In our case, we have:

$$\frac{1}{n+1} < \frac{1}{c_n} < \frac{1}{n}, \text{ so } \frac{1}{n+2} < \frac{1}{1+c_n} < \frac{1}{n+1}$$

It follows that:

$$a_n < \sqrt{n}\left[1 + \frac{1}{c_n}\right]^{c_n}\left[\ln\left(1 + \frac{1}{n}\right) - \frac{1}{n+2}\right] =$$

$$= \frac{\sqrt{n}}{n+2}\left[1 + \frac{1}{c_n}\right]^{c_n}\left[\ln\left(1 + \frac{1}{n}\right)^{n-2} - 1\right] \longrightarrow$$

$$\longrightarrow 0 \cdot e \cdot 0 = 0.$$

(B) The calculation of the limits of sequences that can be put under one of the formulas:

$$1_B.\quad a_n = f'(1) + f'(2) + \ldots + f'(n)$$

$$2_B.\quad a_n = f'(1) + f'(2) + \ldots + f'(n) - f(n),$$

f being a function that is subject to Lagrange's theorem on the intervals: $[n, n+1], n \in \mathbb{N}$.

Observation: **the sequences of the form 2_B are always monotonic and bounded (so convergent) if f is a function**

differentiable on $\mathbb{R}$ and so that f and f' have different monotonies.

Demonstration: to make a choice, let's assume f is increasing and f' is decreasing:

1. the monotony:

$$a_{n+1} - a_n = [f'(1) + f'(2) + \dots + f'(n) - f(n+1)] -$$
$$- [f'(1) + f'(2) + \dots + f'(n) - f(n)] =$$
$$= f'(n+1) - (f(n+1) - f(n))$$

2. we apply Lagrange's theorem to function f on the interval $[n, n+1]$:

$$\exists\ c_n \in (n, n+1) \text{ a.i. } f(n+1) -$$
$$- f(n) = f'(c_n) \qquad (3.7)$$
$$n < c_n < n+1 \Rightarrow f'(n) > f'(c_n) > f'(n+1) \qquad (3.8)$$

so:

$$a_{n+1} - a_n = f'(n+1) - f'(c_n) < 0$$

(f'- decreasing), wherefrom we deduce the sequence is decreasing.

3. the bounding: being decreasing, the sequence is superiorily bounded by a_1. We have only to find the inferior bound. For this, we write (3.8) starting with $n = 1$ and we thus obtain the possibility of minoring (majoring) the sequence a_n.

$$n = 1 \quad 1 < c_1 < 2 \Rightarrow f'(1) > f'(c_1) > f'(2)$$
$$n = 2 \quad 2 < c_2 < 3 \Rightarrow f'(2) > f'(c_2) > f'(3)$$
$$n = 3 \quad 3 < c_3 < 4 \Rightarrow f'(3) > f'(c_3) > f'(4)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$n = n \quad n < c_n < n + 1 \Rightarrow f'(n) > f'(c_n) > f'(n+1)$$

$$\text{so } a_n = f'(1) + f'(2) + \dots + f'(n) - f(n) >$$
$$> f'(c_1) + f'(c_2) + \dots + f'(c_n) - f(n)$$

We obtain a convenient replacement for the sum:

$$f'(c_1) + f'(c_2) + \dots + f'(c_n)$$

We write the relation (3.7) starting $n = 1$ and we add the obtained equalities:

$$n = 1 \quad \exists\, c_1 \in (1, 2) \quad f(2) - f(1) = f'(c_1)$$

$$n = 2 \quad \exists\, c_2 \in (2, 3) \quad f(3) - f(2) = f'(c_2)$$

$$n = 3 \quad \exists\, c_3 \in (3, 4) \quad f(4) - f(3) = f'(c_3)$$

....

$$n = n \quad \exists\, c_n \in (n, n + 1) \; f(n + 1) - f(n) = f'(c_n)$$

$$f(n + 1) - f(1) =$$

$$= f'(c_1) + f'(c_2) + \dots + f'(c_n)$$

it follows that:

$$a_n \le f(n + 1) - f(1) - f(n) < - f(1),$$

consequently, the sequence is inferiorly bounded. It is thus convergent. Its limit is a number between $-f(1)$ and a_1.

Exercises

I. Using Lagrange's theorem, calculate the limits of the sequences:

1. $a_n = n^p \left[a^{\frac{1}{n}} - a^{\frac{1}{n+1}} \right], \; a > 0$

2. $a_n = n^p \cdot \ln \dfrac{n + 1}{n}$

3. $a_n = n^p \left(\operatorname{arctg}(n+1) - \operatorname{arctg} n \right)$

4. $a_n = n^p \left[\dfrac{1}{n^q} - \dfrac{1}{(n + 1)^q} \right]$

5. $a_n = \dfrac{a^{n+1} - a^n}{n^p}$

6. $a_n = \dfrac{\sin\frac{1}{n} - \sin\frac{1}{n+1}}{\ln\frac{n+1}{n}}$

7. $a_n = n^2\left(\dfrac{\operatorname{arctg} n}{n} - \dfrac{\operatorname{arctg}(n+1)}{n+1}\right)$

II. Study the convergence of the sequences:

1. $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n$

(the limit of this sequence is called Euler's constant $c \in (0,1)$)

2. $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$

(the partial sums of the harmonic sequence $\sum_{i=1}^{\infty}\frac{1}{i}$)

3. $a_n = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \ldots + \frac{1}{n^\alpha}$

(the partial sums of the generalized harmonic sequence)

4. $a_n = \sum_{k=2}^{n} \frac{1}{k \cdot \ln k}$

5. $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+k}$

6. $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$

7. $a_n = \frac{1}{2} + \frac{1}{5} + \ldots + \frac{1}{n^2+1} - \operatorname{arctg} n$

Indication: 4. In order to have one of the forms 1_B or 2_B, we determine f, keeping into account that:

$$f'(k) = \frac{1}{k \cdot \ln k}, \text{ so } f'(x) = \frac{1}{x \cdot \ln x}$$

$$\text{and } f(x) = \int \frac{1}{x \cdot \ln x}\,dx = \ln \ln x .$$

We apply for this function the steps from the demonstration at the beginning of this paragraph.

III. Show that:

1. $$1998 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{10^6}} < 1999$$

2. $$2 \cdot 10^k - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{10^{2k}}} < 2 \cdot 10^k - 1$$

3. $$\frac{p}{p-1}\left(a^{k(p-1)} - 1\right) < 1 + \frac{1}{\sqrt[p]{2}} + \frac{1}{\sqrt[p]{3}} + \ldots + \frac{1}{\sqrt{a^{pk}}} < \frac{1}{p-1}\left(a^{k(p-1)} - \frac{1}{p}\right)$$

4. $$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^p} > \frac{n+1}{2}$$

Indication: 1. we must show that:

$$1998 < f'(1) + f'(2) + \ldots f'(10^6) < 1999$$

for:

$$f(x) = \int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} .$$

The harmonic sequence

The sequence $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is called a harmonic sequence because it has the property: any three consecutive terms are in a harmonic progression. Indeed, if we note $b_n = \frac{1}{n}$, we have:

$$b_n = \frac{2}{\frac{1}{b_{n-1}} + \frac{1}{b_{n+1}}}.$$

For a long time, it was believed that this sequence has a defined sum s. In antiquity they sought to obtain the approximate value of s, by calculating the sum of as many terms of the sequence as possible. Today it is a fact that $s = \infty$ (the harmonic sequence is divergent). We mention a few of the methods that help prove this, methods that use exercises from high school manuals, or exercises of high school level:

1. Lagrange's theorem (exercise I.2)

2. The inequality:

$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{13}{24}$$

that is proven by induction in the X grade. Indeed, if s had a finite value:

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots = s < \infty$$

then, by noting:

$$R_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots$$

we would have:

$$a_n + R_n = s$$

and because $s = \lim_{n \to \infty} a_n$, we deduce $\lim_{n \to \infty} R_n = 0$.

But:

$$R_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} +$$

$$+ \frac{1}{2n+1} + \ldots > \frac{1}{n+1} + \frac{1}{n+2} +$$

$$+ \ldots + \frac{1}{2n} > \frac{13}{24}$$

so R_n doesn't tend to zero. Therefore, s isn't finite either.

3. Using the limit studied in the XII grade:

$$\lim_{n \to \infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}\right) = \ln 2$$

4. Using the inequality:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{2^n} > \frac{n+1}{2} .$$

5. Using the definition of the integral (Riemann sums) (see method 16, exercise II).

14. Sequences given by recurrence relations

(A) The linear Recurrence

1. The first degree linear recurrence

Is of the form:

$$a_{n+1} = \alpha \cdot a_n \ , \text{ with } a_0 \text{ given}$$

The expression of the general term follows from the observation that:

$$a_{n+1} = q \cdot a_n = q^2 \cdot a_{n-1} = \dots = q^{n-1} \cdot a_0$$

so:

$$a_{n+1} = q^{n+1} \cdot a_0 \ .$$

Example:

$$a_{n+1} - a_n = \frac{1}{10} a_n \ , \text{ with } a_0 \text{ given.}$$

We have:

$$a_{n+1} = \frac{11}{10} a_n = \left(\frac{11}{10}\right)^{n+1} \cdot a_0 \ .$$

2. Second degree linear recurrence

Is of the form:

$$a_{n+1} = \alpha_1 a_n + \alpha_2 a_{n-1}$$ **, with a_0 and a_1 given.**

In order to find the expression of the general term, in this case, we will use the expression of the general term from the first order recurrence. We saw that for this recurrence we have: $a_n = a_0 q^n$. For the second order recurrence, we search for the general term with the same form:

$a_n = c \cdot q^n$, c being an undetermined constant.

By replacing in the recurrence relation, we obtain:

$$q^{n+1} = \alpha_1 \cdot q^n + \alpha_2 \cdot q^{n-1}$$

from where, dividing by q^{n-1}, we obtain **the characteristic equation**:

$$q^2 - \alpha_1 \cdot q - \alpha_2 = 0$$

We consider the following cases:

a) $\Delta > 0$ (the characteristic equation has real and distinct roots q_1 and q_2)

In this case, having no reason to neglect one of the roots, we modify the expression a_n, considering it of the form:

$$a_n = c_2 \cdot q_1^n + c_2 \cdot q_2^n$$

and we determine the constants c_1 and c_2 so that the first two terms of the sequence have the initially given values.

Example:

$$a_{n+1} = a_n + a_{n-1}$$ **with** $a_0 = a_1 = 1$

(Fibonacci's sequence). The characteristic equation is:

$$q^2 - q - 1 = 0$$

with the roots:

$$q_1 = \frac{1 - \sqrt{5}}{2} \text{ and } q_2 = \frac{1 + \sqrt{5}}{2} .$$

Considering a_n of the form:

$$a_n = c_1\left(\frac{1-\sqrt{5}}{2}\right)^n + c_2\left(\frac{1+\sqrt{5}}{2}\right)^n,$$

from the system:

$$\begin{cases} a_0 = 1 \\ a_1 = 1 \end{cases}$$

we obtain:

$$c_1 = \frac{\sqrt{5}-1}{2\sqrt{5}} \quad \textbf{and} \quad c_2 = \frac{\sqrt{5}+1}{2\sqrt{5}}$$

so:

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \ n \geq 0.$$

Observation: $q_2 = \frac{1+\sqrt{5}}{2}$ is known from antiquity as *the golden ratio*. It is the limit of the sequence:

$$a_n = \sqrt{1 + \sqrt{1 + \ldots + \sqrt{1}}}$$

(n radical) (method 11 applies). This number is often found in nature (the arrangement of branches on trees, the proportion of the human body etc. (For details, see, for example Matila Ghika: *Esthetics and art theory*, Encyclopedic and Scientific Press, Bucharest, 1981).

b) $\Delta = 0$ (the characteristic equation has equal roots $q_1 = q_2$). In this case we consider a_n of the form:

$$a_n = c_1 \cdot q_1^n + n \cdot c_2 \cdot q_1^n$$

(namely $a_n = q_1^n \cdot P_1(n)$, with P_1- first degree polynomial) and we determine $c_1$and c_2, stating the same condition, that the first two terms have the same initially given values.

Example:

$$a_{n+2} = a_{n+1} - \frac{1}{4} a_n \ , \quad a_0 = 1 \ , \quad a_1 = 2 \ .$$

Taking $a_n = c \cdot q^n$, we obtain the characteristic equation:

$$q^2 - q + \frac{1}{4} = 0 \ , \text{ with the roots } q_1 = q_2 = \frac{1}{2} \ .$$

Considering:

$$a_n = \left(\frac{1}{2}\right)^n \cdot (c_1 + c_2 n) \ ,$$

from the system:

$$\begin{cases} a_0 = 1 \\ a_1 = 2 \end{cases} \ ,$$

we obtain: $c_1 = -4$, $c_2 = 6$, so:

$$a_n = \frac{1}{2^n}(6n - 4) \text{ and } \lim_{n \to \infty} a_n = 0 \ .$$

c)) $\Delta < 0$ (the characteristic equation has complex roots)

Let these be:

$$q_1 = r(\cos t + i \cdot \sin t) \text{ and } q_2 = r(\cos t - i \cdot \sin t)$$

We have:

$$a_n = c_1 \cdot q_1^n + c_2 \cdot q_2^n \ ,$$

but, in order to use only natural numbers, we show that we can replace q_1^n and q_z^n respectively with:

$$\frac{q_1^n + q_2^n}{2} = r^n \cos nt \text{ and } \frac{q_1^n - q_2^n}{2} = r^n \sin nt$$

Then we can write:

$$a_n = r^n (c_1 \cos nt + c_2 \sin nt) r.$$

3. Linear recurrence of degree h (bigger than 2)

Is of the form:

$$a_{n+h} = \alpha_1 \cdot a_{n+h-1} + \alpha_2 \cdot a_{n+h-2} + \dots + \alpha_h \cdot a_n$$

with the first h terms given.

Proceeding as with the second order recurrence, we combine $a^n = c \cdot q$ and we state the condition that the recurrence relation be met. We obtain the characteristic equation:

$$q^h = \alpha_1 \cdot q^{h-1} + \alpha_2 \cdot q^{h-2} + \dots + \alpha_h$$

We distinguish the following cases:

a) if the characteristic equation has all the roots real and distinct: $q_1, q_2, \dots, q_h$, we will consider a^n of the form:

$$a_n = c_1 \cdot q_1^n + c_2 \cdot q_2^n + \dots + c_h \cdot q_h^n$$

and we determine the constants c_i, $i \in \overline{1, h}$, stating the condition that the first k terms of the sequence have the initially given values.

b) if a root, for example, q_1 is multiple of order s, ($q_1 = q_2 = \cdots = q_s$), we replace the sum:

$$c_1 \cdot q_1^n + c_2 \cdot q_2^n + \dots + c_s \cdot q_s^n$$

from the expression of a_n with:

$$c_1 \cdot P_{s-1}(n) \quad ,$$

P_{s-1} being a polynomial of degree $s - 2$, whose coefficient is revealed by stating the condition that the first s terms have the initially given values.

c) if a root, for example, q_1, is complex, then its conjugate is also a root of the characteristic equation (let for example, $q_2 = \overline{q_1}$). In this case, we replace in the expression of a_n the sum:

$$c_1 \cdot q_1^n + c_2 \cdot q_2^n$$

with:

$$r^n(c_1 \cos nt + c_2 \sin nt)$$

and if the root q_1 is a multiple of order s ($q_1 = q_2 = \cdots q_s$ and

$$q_{s+1} = q_{s+2} = \dots = q_{2s} = \overline{q_1} \;)$$

we replace in the expression of a_n the terms that contain the complex root with:

$$r^n(\,c_1\cos nt + c_2\sin nt\,) +$$

$$+\, n\cdot r^n(\,c_3\cos nt + c_4\sin nt\,) +$$

$$+ \dots + n^{s-1}\cdot r^n(\,c_{2s-1}\cos nt + c_{2s}\sin nt\,).$$

EXERCISES:

I. Determine the expression of the general term and calculate the limit for:

1. $a_{n+1} = \frac{3}{2} a_n$, **with** a_0 **given**
2. $a_{n+1} = a_n \cdot \ln x$, **with** $a_0 = a^x$
3. $a_{n+1} = a^x \cdot a_n$ **with** $a_0 = \ln x$
4. $a_{n+2} = \dfrac{3\cdot a_{n+1} - a_n}{2}$ **with** $a_1 = 1$, $a_2 = 2$
5. $a_{n+2} = 2\cdot a_{n+1} - 2\cdot a_n$ **with** $a_1 = 1$, $a_2 = 2$
6. $a_{n+3} = 7\cdot a_{n+2} - 16\cdot a_{n+1} + 12\cdot a_n$,
   $a_0 = 0$, $a_1 = 1$, $a_2 = -1$
7. $a_{n+3} = 3\cdot a_{n+2} - 3\cdot a_{n+1} + a_n$,
   $a_0 = 1$, $a_1 = 6$, $a_2 = 17$
8. $a_{n+4} + 2\cdot a_{n+3} + 3\cdot a_{n+2} + 2\cdot a_{n+1} + a_n = 0$,
   $a_0 = a_1 = 0$, $a_2 = -1$, $a_3 = 0$

II. 1. Determine α so that the sequence given by the recurrence relation:

$$a_{n+2} = \alpha \cdot a_{n+1} = 3 \cdot a_n \quad , \quad a_1 = 0 \quad , \quad a_2 = 2$$

has a limit and calculate that limit.

2. Write the recurrence relations and the general terms of the sequences for which: $a_1 = 1$, $a_2 = 2$, and the roots of the characteristic equation are:

a) $q_1 = 1$, $q_2 = -\frac{1}{2}$

b) $q_1 = q_2 = 1$, $q_3 = \frac{1}{2}$

c) $q_1 = \frac{1 + i \cdot \sqrt{3}}{2}$, $q_2 = \frac{1 - i \cdot \sqrt{3}}{2}$

d) $q_1 = 1 + i$, $q_2 = 1 - i$, $q_3 = 1$

3. Let : $a_n = A \cdot \alpha^n + B \cdot \beta^n$, with $a, B, \alpha, \beta \in \mathbb{R}$ and $A, B \neq 0, |\alpha| \neq |\beta|$. Determine $\alpha \ and \ \beta$ so that:

a) the sequence (x_n) is convergent;

b) $\lim_{n \to \infty} x_n = \infty$

c) $\lim_{n \to \infty} x_n = -\infty$

4. Let $a_n = A \cdot \alpha^n + B \cdot n \cdot \beta^n, n \geq 1$, with $a, B, \alpha, \beta \in \mathbb{R}$ and $A, B \neq 0$. Determine $\alpha \ and \ \beta$ so that the sequence is convergent.

(B) Nonlinear recurrence

1. Recurrence of the form $a_{n+1} = \alpha \cdot a_n + \beta$, with a_0 given

The expression of the general term is obtained by observing that if l is a root for the equation $l = \alpha \cdot l + \beta$ (obtained by

replacing a_{n+1} and a_n with l in the recurrence relation), then the sequence $(a_n - l)_{n\in\mathbb{N}}$ is a geometric progression.

Example:

$$a_1 = \frac{1}{2}, \quad a_{n+1} = \frac{a_n + 1}{2}$$

From the equation $l = \frac{l+1}{2}$ we deduce $l = l$.

Then the sequence $(a_n - l)_{n\in\mathbb{N}}$ is a geometric progression.

Let $b_n = a_n - 1$. We have:

$$b_1 = a_1 - 1 = -\frac{1}{2},$$

$$b_2 = a_2 - 1 = \frac{a_1 + 1}{2} - 1 = -\frac{1}{4}.$$

The ratio of the geometric progression is, consequently: $q = \frac{1}{2}$. We obtain:

$$b_n = b_1 \cdot q^{n-1} = -\frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = -\left(\frac{1}{2}\right)^n,$$

$$\text{so } a_n = b_n + 1 = \left(\frac{1}{2}\right)^n + 1 \text{ and } \lim_{n\to\infty} a_n = 1.$$

2. Recurrence of the form $a_{n+1} = \alpha \cdot a_n + f(n)$ (f being a random function)

The general term is found based on the observation that, if $(b_n)_{n\in\mathbb{N}}$ is a sequence that verifies the same recurrence relation, then any sequence $(\alpha_n)_{n\in\mathbb{N}}$ verifies the given relation of the form:

$$a_n = c \cdot \alpha^n + b_n, \text{ with } c = a_0 - b_0$$

In practice we chose b_n of the form:

$$b_n = f(n)$$

the coefficients of f being undetermined. We determine these coefficients by stating the condition that $(b_n)_{n\in\mathbb{N}}$ verifies the recurrence relation.

Particular case: a recurrence with the form:

$$a_{n+1} = \alpha \cdot a_n + \beta^n \cdot P(n)$$

with P a polynomial of degree s.

Example:

$$a_{n+1} = 3 \cdot a_n + 5^n (n^2 + 2n + 3), \quad a_0 = 1$$

We search for the sequence b_n with the form:

$$b_n = 5^n (u \cdot n^2 + v \cdot n + w) .$$

We state the condition that $(b_n)_{n\in\mathbb{N}}$ verifies the given recurrence relation:

$$b_{n+1} = 3 \cdot b_n + 5^n (n^2 + 2n + 3)$$

We obtain:

$$5^{n+1} (u(n + 1)^2 + v(n + 1) + w) =$$
$$= 3 \cdot 5^n (un^2 + vn + w) + 5^n (n^2 + 2n + 3)$$

By identification, it follows that:

$$a = \frac{1}{2}, \quad v = -\frac{3}{2}, \quad w = \frac{21}{4},$$

so:

$$b_0 = \frac{21}{4} \quad \textbf{and} \quad a_n = 5^n (a_0 - b_0) + b_n =$$
$$= -\frac{17}{4} \cdot 5^n + 5^n \left(\frac{1}{2} u^2 - \frac{3}{2} u + \frac{21}{4} \right) =$$
$$= 5^n \left(\frac{1}{2} n^2 - \frac{3}{2} n + 1 \right) .$$

3. Recurrence of the form $a_{n+1} = f(a_n)$ (with $f: [a, b] \rightarrow \mathbb{R}$ continuous function)

Theorem: 1. If the function $f: [a, b] \rightarrow \mathbb{R}$ is continuous and increasing on $[a, b]$, then:
a) if $a_1 > a_0$, the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is increasing,
b) if $a_1 < a_0$, the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is decreasing.
The limit of the sequence is a fixed point of f, i.e. a characteristic equation $f(x) = x$.
2. If f is decreasing on $[a, b]$, then the subsequences of even and uneven value, respectively, of $(\alpha_n)_{n\in\mathbb{N}}$ are monotonic and of different monotonies. If these two subsequences have limits, then they are equal.

Example:

$$a_1 = 10 \ , \quad a_{n+1} = \frac{2 + a_n^2}{2 \cdot a_n}$$

We have:

$$f(x) = \frac{2 + x^2}{2x} \ ;$$

from the variation table we deduce that f is decreasing on $(0, \sqrt{2})$ and increasing on $(\sqrt{2}, \infty)$. By way of induction, we prove that $\alpha_n > \sqrt{2}$, and, because $a_2 = 5{,}1 < a_1$, then the sequence is decreasing.

The bounding can be deduced from the property of continuous functions defined on closed intervals and bounded of being bounded.

The characteristic equation $f(x) = x$ has the roots $x_{1,2} = \pm\sqrt{2}$, so $l = \sqrt{2}$.

Particular case: the homographic recurrence:

$$a_{n+1} = \frac{\alpha \cdot a_n + \beta}{\gamma \cdot a_n + \delta}, \text{ with } a_0 \text{ given}$$

We have:

$$f(x) = \frac{\alpha \cdot x + \beta}{\gamma \cdot x + \delta},$$

called a homographic function and:

$$f'(x) = \frac{\alpha \cdot \delta - \beta \cdot \gamma}{(\gamma \cdot x + \delta)^2},$$

so the monotony of f depends on the sign of the expression $\alpha \cdot \delta - \beta \cdot \gamma$.

If $x_1 \neq x_2$ are the roots of the characteristic equation $f(x) = x$, by noting:

$$\theta = \frac{\gamma \cdot x_1 + \delta}{\gamma \cdot x_2 + \delta}$$

it can be shown that the general term α_n of the sequence is given by the relation:

$$\frac{x_2 - a_n}{x_1 - a_n} = \theta^n \frac{x_2 - a_0}{x_1 - a_0}$$

Example:

$$a_0 = 1, \quad a_{n+1} = \frac{2}{1 + a_n}.$$

Prove that:

$$a_1 < a_3 < a_5 < \ldots < 1 < \ldots < a_4 < a_2 < a_0.$$

From here it follows that the sequence is convergent (it decomposes in two convergent sequences). It is deduced that the limit of the sequence is 1.

Exercises

I. Determine the general term and study the convergence of the sequences:

1. $a_1 = 1$, $a_{n+1} = a \cdot a_n + 1$ with $|a| < 1$

2. $a_0 = \frac{1}{2}$, $a_{n+1} = 2a_n + 3^n(n^2 - n - 1)$

3. $a_0 = 0$, $a_{n+1} = -\frac{1}{2} a_n + \frac{3}{2^n} \cdot n$

4. $a_0 = 0$, $a_{n+1} = \frac{1}{2} a_n + \frac{n^2}{2^n}$

5. $a_0 = 1$, $a_{n+1} = \dfrac{a_n}{\sqrt{a_n^2 + 1}}$

6. $a_0 > 1$, $a_{n+1} = \dfrac{2}{1 + a_n}$

7. $a_0 = 1$, $a_{n+1} = \dfrac{2 \cdot a_n - 1}{2 \cdot a_n + 5}$

8. $a_0 = \frac{1}{2}$, $a_{n+1} = a \cdot a_n + b$.

Determine coefficients a and b so that the sequence has a finite limit. Calculate this limit.

9. $a_1 \neq 2$, $a_{n+1} = \dfrac{2}{2 - a_n}$ is periodic

II. 1. $x_0 \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right)$;

$\sin x_n + \cos x_n = 1$

2. $x_0 = a$, $x_{n+1} = \dfrac{x_n^3 + 4 \cdot x_n}{8}$

3. $x_0 = 1$, $x_{n+1} = \frac{1}{2}\left[x_{n-1} + \dfrac{a}{x_{n-1}} \right]$

4. $x_0 = 1$, $x_{n+1} = \sqrt{\dfrac{x_n}{a + b \cdot x_n}}$, $a, b > 0$.

15. Any Cauchy sequence of natural numbers is convergent

One of the recapitulative exercises from the XI grade analysis manual requires to be shown that if a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of real numbers is convergent, then:

$$\forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \quad \forall m,n \geq \varepsilon \quad |a_m - a_n| < \varepsilon \qquad (3.9)$$

A sequence that satisfies condition (3.8) is called a Cauchy sequence or a fundamental sequence. It is proven that a natural numbers' sequence is convergent if and only if it is a Cauchy sequence. This propositions enables the demonstration of the convergence of sequences, showing that they are Cauchy sequences.

In this type of exercises the following condition is used:

$$\forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \quad \forall m,n \geq n_\varepsilon \quad \forall p \in \mathbb{N}$$

$$|a_{n+p} - a_n| < \varepsilon \qquad (3.9')$$

which is equivalent to (3.9) but easier to use.

Example:

$$a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$$

It has to be proven that:

$$\text{(a)} \ \forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \quad \forall n \geq n_\varepsilon \quad \forall p \in \mathbb{N}$$

$$|a_{n+p} - a_n| < \varepsilon$$

(b) let $\varepsilon > 0$. We check to see if n_ε exists so that property (a) takes place.

(c) we have:

$$|a_{n+p} - a_n| = \frac{1}{(n+p)!} + \frac{1}{(n+p-1)!} +$$

$$+ \ldots + \frac{1}{(n+1)!} =$$

$$= \frac{1}{n!}\left(\frac{1}{(n+1)(n+2)\ldots(n+p)} + \ldots + \frac{1}{n+1}\right) <$$

$$< \frac{1}{n!}\left[\left(\frac{1}{2}\right)^{p} + \left(\frac{1}{2}\right)^{p-1} + \dots + \frac{1}{2}\right] =$$

$$\frac{1}{n!}\cdot\frac{1-\left(\frac{1}{2}\right)^{p}}{1-\frac{1}{2}} < \frac{1}{n!}\cdot\frac{1}{\frac{1}{2}} = \frac{2}{n!} < \frac{2}{n}$$

(d) we state the condition: $\frac{2}{n} < \varepsilon$ and we obtain $n >: \frac{2}{n}$, so we can take: $n_\varepsilon = \left[\frac{2}{n}\right] + 1$.

Exercises

I. Show that the following sequences are fundamental:

1. $a_n = \frac{1}{n}$

2. $a_n = \frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \dots + \frac{1}{n(n+2)}$

3. $a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}$

4. $a_n = 1 + \frac{\cos x}{3} + \frac{\cos 2x}{3^2} + \dots + \frac{\cos nx}{3^n}$

5. $a_n = \frac{\sin x}{n^2+1} + \frac{\sin 2x}{n^2+2} + \dots + \frac{\sin nx}{n^2+n}$

6. $a_n = \frac{\cos x}{2} + \frac{\cos 2x}{2^2} + \dots + \frac{\cos nx}{2^n}$

7. $a_n = \frac{\cos \alpha_1}{1\cdot 2} + \frac{\cos \alpha_2}{2\cdot 3} + \dots + \frac{\cos \alpha_n}{n(n+1)}$

8. $a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + (-1)^{n-1}\frac{1}{n}$

9. **Let** $a_n = b_1 + b_2 + \dots + b_n$ **where:**

a) $b_k \cdot b_{k+1} < 0$

b) $|b_1| > |b_2| > \dots > |b_n| > \dots$

c) $\lim_{n \to \infty} b_n = 0$

Show that $|a_{n+p} - a_n| < b_{n+1}$ for any $n, p \in \mathbb{N}$. Deduce from this that the sequence is a Cauchy sequence.

INDICATIONS:

3. $|a_{n+p} - a_n| < \frac{1}{n(n+1)} +$

$$+ \frac{1}{(n+1)(n+2)} + \dots + \frac{1}{(n+p-1)(n+p)}$$

and we decompose in simple fractions.

8. $|a_{n+p} - a_n| = \left| (-1)^{n+p-1} \frac{1}{n+p} + \right.$

$$\left. + (-1)^{n+p-2} \frac{1}{n+p-1} + \dots + (-1)^n \frac{1}{n+1} \right| =$$

$$= \left| \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \dots + (-1)^{p-1} \frac{1}{n+p} \right|$$

If p is uneven, we deduce:

$$|a_{n+p} - a_n| = \left| \frac{1}{n+1} - \left(\frac{1}{n+2} - \frac{1}{n+3} \right) - \right.$$

$$\left. - \dots - \left(\frac{1}{n+p-1} - \frac{1}{n+p} \right) \right| < \frac{1}{n+1},$$

and if it is even:

$$|a_{n+p} - a_n| = \left| \frac{1}{n+1} - \left(\frac{1}{n+2} - \frac{1}{n+3} \right) - \right.$$

$$\left. - \dots - \left(\frac{1}{n+p-2} - \frac{1}{n+p-1} \right) - \frac{1}{n+p} \right| < \frac{1}{n+1}$$

So:

$$|a_{n+p} - a_n| < \frac{1}{n+1}.$$

From the condition: $\frac{1}{n+1} < \varepsilon$, we obtain: $n_\varepsilon = \left[\frac{1}{\varepsilon} - 1\right] + 1$.

9. We proceed as we did in the previous exercise, where we had:

$$b_n = (-1)^n \frac{1}{n} \quad .$$

16. Using the definition of the integral

It is known that the Riemann sum attached to the function $f: [a, b] \to \mathbb{R}$, corresponding to a division:

$$\Delta = \{ a = x_0 < x_1 < x_2 < \ldots < x_n = b \}$$

and to the intermediate points:

$$\xi_i \in [x_{i-1}, x_i] \ ,$$

is:

$$\sigma_\Delta(f, \xi) = \sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1}) \qquad (3.10)$$

If the points x_i are equidistant, then:

$$x_i - x_{i-1} = \frac{b-a}{n}$$

and we have:

$$\sigma_\Delta(f, \xi) =$$

$$= \frac{b-a}{n} \left[f(\xi_1) + f(\xi_2) + \ldots + f(\xi_n) \right] \quad (3.11)$$

A function is integrable if:

$$\lim_{\|\Delta\| \to 0} \sigma_\Delta(f, \xi)$$

exists and is finite. The value of the limit is called the integral of function f on the interval $[a, b]$:

$$\lim_{\|\Delta\| \to 0} \sigma_\Delta(f, \xi) = \int_a^b f(x) \, dx$$

To use the definition of the integral in the calculation of the limits of sequences, we observe that for divisions Δ formed with equidistant points, we have:

$$\lim_{\|\Delta\| \to 0} \sigma_\Delta(f,\xi) = \lim_{n \to \infty} \sigma_\Delta(f,\xi)$$

So we can proceed as follows:

a) we show that the general term a_n can be written in the form:

$$a_n = \frac{b-a}{n}\left[f(\xi_1) + f(\xi_2) + \ldots + f(\xi_n)\right]$$

with f as a continuous function on $[a,b]$, and ξ_i being the points of an equidistant division.

b) f' being continuous, it is also integrable, and:

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \sigma_\Delta(f,\xi) \overset{\Delta \text{ has equidistant}}{\underset{\text{points}}{=\!=\!=\!=\!=\!=}}$$

$$\lim_{\|\Delta\| \to 0} \sigma_\Delta(f,\xi) \overset{f\text{ - continuous}}{=\!=\!=\!=\!=} \int_a^b f(x)\,dx.$$

Example:

$$a_n = n^2 \sum_{i=1}^{n} \frac{1}{n^3 + k^3}$$

a) we write a_n in the form:

$$a_n = \frac{b-a}{n}\left[f(\xi_1) + f(\xi_2) + \ldots + f(\xi_n)\right]$$

highlighting the common factor $\frac{1}{n}$. We have:

$$a_n = \frac{1}{n} \sum_{i=1}^{n} \frac{n^3}{n^3 + k^3} =$$

$$= \frac{1}{n}\left(\frac{n^3}{n^3 + 1^3} + \frac{n^3}{n^3 + 2^3} + \ldots + \frac{n^3}{n^3 + n^3}\right)$$

b) in:

$$f(\xi_i) = \frac{n^3}{n^3 + i^3}$$

we make the equidistant points: $\frac{(b-a)i}{n}$ appear:

$$\frac{n^3}{n^3 + i^3} = \frac{1}{1 + \left(\frac{i}{n}\right)^3}$$

and we deduce the function f. We have:

$$f(x) = \frac{1}{1 + x^3}.$$

c) we arrange the equidistant points on a straight line:

and we deduce the division:

$$\Delta = \left\{ 0, \ldots, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1 \right\}$$

and the interval $[a, b]$.

d) we observe that a_n is the Riemann sum corresponding to function f (continuous, thus integrable) and to the deduced division Δ, on interval $[a, b]$. As the points of Δ are equidistant, we have:

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \sigma_\Delta(f, \xi) = \lim_{\|\Delta\| \to 0} \sigma_\Delta(f, \xi) =$$

$$= \int_0^1 \frac{1}{1 + x^3}\, dx = \frac{\sqrt{3}}{3}.$$

Exercises:

I. Using the definition of the integral, calculate the limits of the following sequences:

1. $a_n = \frac{1}{n^6} \sum_{i=1}^{n} i^5$

2. $a_n = n \sum_{i=1}^{n} \frac{1}{3n^2 + i^2}$

3. $a_n = \frac{1}{n^2} \sum_{i=1}^{n} i.e^{i/n}$

4. $a_n = \frac{\pi}{2n} \sum_{i=1}^{n-1} \cos \frac{i\pi}{2n}$

5. $a_n = \sum_{i=1}^{n} \frac{n}{(n+i)\sqrt{n^2+i^2}}$

6. $a_n = \frac{1}{n} \sqrt[n]{n(n+1)(n+2)\ldots(n+n)}$

7. $a_n = \frac{1}{n}\left(\sum_{i=1}^{n} \ln(i^2+n^2) - 2(n-1)\ln n\right)$

Indications:

4. $a_n = \frac{\pi/2}{n} \sum_{i=1}^{n} \cos \frac{(\pi/2)i}{n}$

$$\xrightarrow{n \longrightarrow \infty} \int_0^{\pi/2} \cos x \, dx.$$

6. By making logarithms, we obtain:

$$\ln a_n = -\ln n + \frac{1}{n}(\ln n + \ln(n+1) + \ldots +$$

$$\ln(n+n)) = -\ln n +$$

$$+\frac{1}{n}(\ln n + \ln n(1+\frac{1}{n}) + \ldots + \ln n(1+\frac{n}{n})) =$$

$$= \frac{1}{n}(\ln(1+\frac{1}{n}) + \ln(1+\frac{2}{n}) + \ldots +$$

$$+ \ln(1+\frac{n}{n})) \xrightarrow{n \longrightarrow \infty} \int_0^1 \ln(1+x)\, dx.$$

II. a) Write the Riemann sum

$\sigma_\Delta(f,\xi)$

corresponding to the function:

$$f : [\varepsilon, 1] \longrightarrow \mathbb{R}, \quad f(x) = \frac{1}{x},$$

the division:

$$\Delta = (\varepsilon, \frac{1}{n}, \frac{2}{n}, \ldots \frac{n}{n} = 1)$$

and the intermediate points:

$\xi_i = i/n.$(b)

b) Calculate:

$$\lim_{n \to \infty} \sigma_\Delta(f, \xi) = l(\varepsilon).$$

c) Calculate:

$$\lim_{\varepsilon \to 0} l(\varepsilon)$$

and deduce that:

$$\lim_{n \to \infty} (1 + \frac{1}{2} + \ldots + \frac{1}{n}) = \infty.$$

IV. Continuity and derivability

Continuity

DEFINITION: the function $f: D \to \mathbb{R}$ is continuous in $x_0 \in D$ if:

1. it has a limit in $x_0 \in$,
2. the limit is equal to $f(x_0)$,

i.e.

$$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 \quad \forall x \in D \quad |x - x_0| < \delta_\varepsilon \Rightarrow$$

$$\Rightarrow \quad |f(x) - f(x_0)| < \varepsilon$$

THE CONSEQUENCE: Any continuous function in a point has a limit in that point.

(A) Methods for the study of continuity

1. Using the definition

Example:

$$f(x) = \frac{x + 1}{2x + 3}$$

is continuous in $x = 1$.

Indeed, let's show that:

$$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 \quad \forall x \in \mathbb{R} \setminus \{-\frac{3}{2}\}$$

$$|x - 1| < \delta_\varepsilon \Rightarrow \left| \frac{x + 1}{2x + 3} - f(1) \right| < \varepsilon$$

Let there be any $\varepsilon > 0$. We must determine δ_ε. We have:

$$\left| \frac{x + 1}{2x + 3} - f(1) \right| = \left| \frac{x + 1}{2x + 3} - \frac{2}{5} \right| =$$

$$= \frac{|x - 1|}{5|2x - 3|} < \frac{\delta_\varepsilon}{5|2x + 3|} < \frac{\delta_\varepsilon}{15}$$

because on the interval $[0,2]$, for example, $[2x+3]$ is bounded between 3 and 7. We determine δ_ε from the condition:

$$\frac{\delta_\varepsilon}{15} < \varepsilon .$$

We can take, for example $\delta_\varepsilon = 2\varepsilon$. Then:

$$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon = 2\varepsilon \quad \forall x \in \mathbb{R} \setminus \{\frac{-3}{2}\}$$

$$|x - 1| < 2\varepsilon \quad \Rightarrow |f(x) - f(1)| < \varepsilon$$

therefore f is continuous in $x = 1$.

2. Using the criterion with the lateral limits

***f* is continuous in**:

$$x_0 \Leftrightarrow l_s(x_0) = l_d(x_0) = f(x_0)$$

where

$$l_s(x_0) \quad \textbf{and} \quad l_d(x_0)$$

are the left and right limit, respectively in x_0.

EXAMPLE: Let's study the continuity of the function:

$$f(x) = \begin{cases} \sqrt{\sin^2\alpha - \frac{x}{e^{1-x}} \sin\alpha + \frac{x^2}{4}} & x \in [0, 1] \\ 1/2 & x = 1 \\ \frac{3}{2(x^2+x+1)} & x \in [1, 2] \end{cases}$$

for $\alpha \in [0, 2\pi]$.

SOLUTION: Because we are required to study the continuity, with a certain point being specified, we must make the study on the whole definition domain. The points of the domain appear to belong to two categories:

1. Connection points between branches, in which continuity can be studied using the lateral limits.

2. The other points, in which the function is continuous, being expressed by continuous functions (elementary) (but this must be specified each time).

In our case:

(a) in any point $x_0 \neq 1$, the function is continuous being expressed through continuous functions.

(b) we study the continuity in $x_0 \neq 1$. We have:

$$l_s(1) = \lim_{\substack{x \to 1 \\ x < 1}} f(x) =$$

$$= \lim_{\substack{x \to 1 \\ x < 1}} \sqrt{\sin^2\alpha - \frac{x}{x^{1-x}} + \frac{x^2}{4}} =$$

$$\lim_{\substack{x \to 1 \\ x < 1}} \sqrt{\left(\sin\alpha - \frac{1}{2}\right)^2} = \left|\sin\alpha - \frac{1}{2}\right|,$$

and $l_d(1) = \lim_{\substack{x \to 1 \\ x < 1}} f(x) = \lim_{x \to 1} \frac{3}{2}(x^2 + x + 1) = \frac{1}{2}.$

The value of the function in the point is $f(1) = \frac{1}{2}$, so f is continuous in:

$$x = 1 \iff \left|\sin\alpha - \frac{1}{2}\right| = \frac{1}{2}.$$

To explain the module we use the table with the sign of the function $\sin\alpha - \frac{1}{2}$:

α	0		$\pi/6$		$5\pi/6$		2π
$\sin\alpha - 1/2$		$-$	0	$+$	0	$-$	

1. if $\alpha \in \left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, 2\pi\right]$, the equation becomes:

$$-\sin\alpha + \frac{1}{2} = \frac{1}{2} \iff \sin\alpha = 0 \Rightarrow$$

$$\Rightarrow \alpha = 0,\ \alpha = \pi,\ \alpha = 2\pi$$

2. if $\alpha \in \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$, we obtain $\sin\alpha = 1$, so $\alpha = \frac{\pi}{2}$.

For this four values of α the function is continuous also in point $x = 1$, so it is continuous on $\mathbb{R}$.

3. Using the criterion with sequences

This criterion is obtained from the criterion with sequences for the limits of functions, by replacing l with $f(x_0)$ and eliminating the condition $x_n \neq x_0$ (which is essential in the case of limits, because we can study the limit of a function in a point where $f(x)$ doesn't exist).

EXAMPLE:

$$f(x) = \begin{cases} 3x^2 + 2x & \text{if } x \in \mathbb{Q} \\ x + 4 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Proceeding as we did with Method 10, with the two adaptations mentioned just above, it follows that f is continuous in the points x_0 for which:

$$3x_0^2 + 2x_0 = x_0 + 4 \text{ , i.e. } x_0 = \frac{-4}{3} \text{ and } x_0 = 1 \text{ .}$$

(B) Types of discontinuity points

The point $x_0 \in D$ in which f is not continuous is said to be a first order discontinuity point if the lateral limits in x_0 exist and are finite.

Any other discontinuity point is said to be of a second order.

(C) The extension through continuity

To extend the function $f: D \to \mathbb{R}$ means to add new points to the domain D, where we define the correspondence law willingly.

If M is the set of added points and h is the correspondence law on M, the extension will be:

$$f_p : D \cup M \longrightarrow \mathbb{R} \text{ , } f_p(x) = \begin{cases} f(x) & \text{if } x \in D \\ h(x) & \text{if } x \in M \end{cases}$$

For example:

$$f : \mathbb{R} \setminus \{ 0 \} \longrightarrow \mathbb{R}, \quad f(x) = \frac{\sin x}{x}$$

can be extended to the entire set ℝ by putting:

$$f_p(x) = \begin{cases} \frac{\sin x}{x} & \text{if} \quad x \in \mathbb{R} \setminus \{0\} \\ \alpha & \text{if} \quad x = 0 \end{cases}$$

There are many extensions to a function. Nevertheless, if f is continuous on D and has a finite limit in a point $x_0 \in D$, there is **only one extension:**

$$f_p : D \cup \{ x_0 \} \longrightarrow \mathbb{R}$$

that is continuous, named the extension through continuity of f in point x_0.

So, for our example, because $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$, the function:

$$f_p(x) = \begin{cases} \frac{\sin x}{x} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

is the only extension of $\sin x / x$ that is continuous on ℝ.

(D) The continuity of composite functions

If f is continuous in x_0 and g is continuous in $f(x_0)$, then $g \circ f$ is continuous in x_0.

$$\begin{array}{ccccc} D & & E & & F \\ x_0 & \longrightarrow & f(x_0) & \longrightarrow & g(f(x_0)) = (gof)(x_0) \end{array}$$

So the composition of two continuous functions is a continuous function. Reciprocally it isn't true: it is possible that f and g are not continuous and $g \circ f$ is.

For example:

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

isn't continuous in any point, but the function $g(x) = (f \circ f)(x)$ is the identically equal function to 1, so it is continuous in any point from $\mathbb{R}$.

Exercises

I. Study, on the maximum definition domain, the continuity of the functions defined by:

1. $f(x) = x \cdot [x]$

2. $f(x) = [x] \cdot \sin \pi x$

3. $f(x) = \lim\limits_{n \to \infty} \dfrac{x^n + x}{x^{2n} + 1}$

4. $f(x) = \lim\limits_{n \to \infty} \dfrac{nx}{1 + |nx|}$

5. $f(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ 2x & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

6. $f(x) = \begin{cases} u(x) & \text{if} \quad x \in \mathbb{Q} \\ v(x) & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

7. $f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \text{ or } x = 0 \\ \dfrac{1}{q} & \text{if} \quad x = \dfrac{p}{q} \end{cases}$

   (Riemann's function)

8. $f(x) = \begin{cases} \left| \dfrac{x - 1}{x + 1} \right| \cdot e^{\frac{-1}{|x|}} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$

9. $$f(x) = \begin{cases} x - p & \text{if } x \in \left[p, p + \frac{1}{2}\right] \quad p \in \mathbb{Z} \\ p + 1 - x & \text{if } x \in \left(p + \frac{1}{2}, p + 1\right] \end{cases}$$

(Manual)

10. $$f(x) = \begin{cases} 27 \ln(-1-x) & \text{if } x < 0 \\ m & \text{if } x = 0 \\ 1+e^{-2x} & \text{if } x > 0 \end{cases}$$

II. Specify the type of discontinuity points for the functions:

1. $$f(x) = \begin{cases} \dfrac{\operatorname{tg} x}{x} & x \neq 0 \\ \alpha & x = 0 \end{cases}$$

2. $$f(x) = \begin{cases} \dfrac{\sqrt{1+x} - 1}{x} & x \neq 0 \\ \alpha & x = 0 \end{cases}$$

3. $$f(x) = \begin{cases} \dfrac{\sin^2 x}{1 - \cos x} & x \neq 2k\pi \\ \alpha & x = 2k\pi \end{cases}$$

4. $$f(x) = \begin{cases} \dfrac{1 - \sin\left(\frac{n\pi}{2}\right)}{x^2 - 8x + 7} & x \neq 1,\ x \neq 7 \\ \alpha & x = 1 \end{cases}$$

5. $$f(x) = \begin{cases} \dfrac{\ln|x|}{1 + \ln|x|} & x \in \mathbb{R} \setminus \left\{-\frac{1}{e}, 0, \frac{1}{e}\right\} \\ \alpha & x \in \left\{-\frac{1}{e}, 0, \frac{1}{e}\right\} \end{cases}$$

Determine the extensions using the continuity, where possible.

III. Study the continuity of the functions $f, g, f \circ g, g \circ f$, for:

1. $$f(x) = \operatorname{sgn} x \qquad g(x) = \begin{cases} x & x \in \mathbb{Q} \\ -x & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

2. $f(x) = [x] \quad g(x) = \ln x$

3. $f(x) = \left[\, 1 + [x] \,\right] \quad g(x) = \operatorname{sgn} x$

4. $f(x) = \operatorname{sgn}(\operatorname{sgn} x) \quad g(x) = \begin{cases} x & x \in \mathbb{Q} \\ 3x^2 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

IV. 1. Let $f, g: \mathbb{R} \to \mathbb{R}$ be a continuous function so that $f(x) = g(x)$ for any $x \in \mathbb{Q}$. Show that $f = g$. *(Manual)*

2. Let $f: I \to \mathbb{R}$, $I \subseteq\to \mathbb{R}$ be a function with Darboux's property. If f has lateral limits in any point I, then f is continuous on I. *(Manual)*

3. $f: \mathbb{R} \to \mathbb{R}$ so that:

$$|f(x) - f(y)| \leq \sqrt{|x-y|}$$

for any $x, y \in \mathbb{R}$. Prove that $a > 0$ exists so that for any x with $|x| \leq a$ we have $|f(x)| < a$. Deduce the existence of a fixed point for f. (x_0 is a fixed point for f if $f(x_0) = x_0$). *(Manual)*

4. Let $f: [a, b] \to \mathbb{R}$, continuous. Then:

$$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 \quad \forall x, y \in [a,b], |x-y| < \delta_\varepsilon \Rightarrow$$

$$\Rightarrow |f(x)-f(y)| < \varepsilon \qquad (4.1)$$

(Manual)

A function with the property (4.1) is **called an uniform continuously function on $[a, b]$.**

The uniform continuity is therefore defined on an interval, while continuity can be defined in a point.

Taking $y = x_0$ in (4.1) it is deduced that any uniform continuous function on an interval is continuous on that interval.

Exercise 4 from above affirms the reciprocal of this proposition, which is true if the interval is closed and bounded.

Derivability

(A) Definition. Geometric interpretation. Consequences

Definition

The function $f: D \to \mathbb{R}$ is differentiable in $x_0 \in D$ if the following limit exists and is finite:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

The value of this limit is noted by $f'(x_0)$ and it is called the derivative of f in x_0.

Geometric interpretation

The derivative of a function in a point x_0 **is the slope of the tangent to the graphic of f in the abscissa point x_0**.

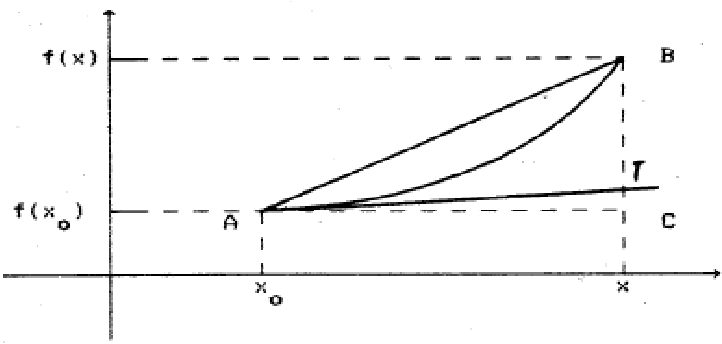


Fig. 4.1

When x tends to x_0, the chord AB tends towards the tangent in A to the graphic, so the slope of the chord

$$m_{AB} = \frac{f(x) - f(x_0)}{x - x_0}$$

tends to the slope of the tangent. Therefore, $f'(x_0)$ is the slope of the tangent in the abscissa point x_0 to the graphic.

The geometric interpretation of the derivative thus follows from the next sequence of implications:

$$AB \xrightarrow{x \to x_0} AT \Rightarrow m_{AB} \xrightarrow{x \to x_0} m_{AT} \Rightarrow$$

$$\Rightarrow \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = m_{AT} \Rightarrow f'(x_0) = m_{AT}$$

Consequences

1. Any function differentiable in a point is continuous in that point.

2. The equation of the tangent to the graphic of a differentiable function.

It is known that equation of a straight line that passes through the point $(x_0, f(x_0))$ is:

$$y - f(x_0) = m(x - x_0)$$

3. The equation of the normal (the perpendicular on the tangent) is deduced from the condition of perpendicularity of two straight lines $(m_1 \cdot m_2 = -1)$ and it is:

$$y - f(x_0) = \frac{-1}{f'(x_0)}(x - x_0)$$

(B) The derivation of composite functions

$$(f(u(x)))' = f'(u(x)) \cdot u'(x) \qquad (4.2)$$

From formula (4.2) it is deduced that:

$$(f_3(f_2(f_1(x))))' = f_3'(f_2(f_1(x))) \cdot f_2'(f_1(x)) \cdot f_1'(x)$$

So, in order to derive a composite function we will proceed as follows:

(a) we derive the last function that is composed (in our case f_3) and we replace in this function the variable x with the expression issuing from the composition of the other function (in our case $f_2 \circ f_1$).

(b) by neglecting the derived function in the previous step (in our case f_3), we derive the function that became last (for us f_2) and we also replace the variable x with the expression that resulted from the composition of the functions that have no yet been derived (in this step we only have $f_1(x)$.

(c) we continue this procedure until all the functions are derived.

EXAMPLE:

$$\cos^2 \sin \sqrt[3]{x^2+1}$$

1. By applying formula (4.2) we have:

$$f'(x) = 2 \cdot \cos \sin \sqrt[3]{x^2+1} \left[\cos \sin \sqrt[3]{x^2+1} \right]' =$$

$$2 \cdot \cos \sin \sqrt[3]{x^2+1} \cdot \left(-\sin \sin \sqrt[3]{x^2+1} \right)$$

$$\left(\sin \sqrt[3]{x^2+1} \right) = 2 \cdot \cos \sin \sqrt[3]{x^2+1} \cdot$$

$$\cdot \left(-\sin \sin \sqrt{x^2+1} \right) \cdot \cos \sqrt{x^2+1} \cdot \left(\sqrt[3]{x^2+1} \right)' =$$

$$2 \cdot \cos \sin \sqrt[3]{x^2+1} \cdot \left(-\sin \sin \sqrt[3]{x^2+1} \right) \cdot$$

$$\cdot \cos \sqrt[3]{x^2+1} \; \frac{1}{3 \cdot \sqrt[3]{(x^2+1)^2}} \cdot 2x$$

We can obtain the same results much faster, using the generalization of formula (4.2) presented above.

Therefore, the composite functions are:

$$f_1(x) = x^2+1 \;,\; f_2(x) = \sqrt[3]{x} \;,\; f_3(x) = \sin x$$

$$f_4(x) = \cos x \ , \ f_5(x) = x^2$$

and $f(x) = f_5(f_4(f_3(f_2(f_1(x)))))$, **so:**

$$f'(x) = f_5'(f_4(f_3(f_2(f_1(x))))) \cdot$$

$$\cdot f_4'(f_3(f_2(f_1(x)))) \cdot f_3'(f_2(f_1(x))) \cdot f_2'(f_1(x)) \cdot f_1'(x) =$$

$$= 2 \cdot \cos \sin \sqrt[3]{x^2 + 1} \cdot \left(-\sin \sin \sqrt[3]{x^2 + 1} \right) \cdot$$

$$\cdot \cos \sqrt[3]{x^2 + 1} \cdot \frac{1}{3 \cdot \sqrt[3]{x^2 + 1}} \cdot 2x$$

CONSEQUENCES:

1. **The derivative of the inverse function.**

From $f^{-1}\big(f(x)\big) = x$, by applying formula (4.2), it follows that:

$$\left(f^{-1}(f(x))\right) \cdot f'(x) = 1 \ ,$$

or by noting $y = f(x)$:

$$\left(f^{-1}(y)\right) = (f'(x))^{-1}$$

EXAMPLE: For $f(x) = \ln x$, $f\colon (0, \infty) \to \mathbb{R}$ we have:

$$f^{-1}(x) = e^x$$

and as $y = \ln x => x = e^y$ we obtain:

$$\left(e^y \right)' = \left((\ln x)'\right)^{-1} = \left(\frac{1}{x}\right)^{-1} = x = e^y.$$

2. High order derivatives of composite functions.

From $\big(f\big(u(x)\big) = f'(u(x)).u'(x)$, deriving again in relation to x it follows that:

$$\left(f(u(x))\right)'' = \left(f'(u(x) \cdot u'(x)\right) =$$

$$= f''(u(x)) \cdot u'(x) \cdot u'(x) +$$

$$+ f'(u(x)) \cdot u''(x) = f''(u(x)) \cdot \left(u'(x)\right)^2 +$$

$$+ f'(u(x)) \cdot u''(x)$$

We have therefore obtained the second order derivative of the composite function and the procedure can go on.

EXAMPLE: Let's calculate $f''(x)$ if $f(x) = g(g^{-x})$, g being a function two times differentiable on ℝ.

We have:

$$f'(x) = g'(e^{-x}) \cdot (-e^{-x})$$

and

$$f''(x) = g''(e^{-x})(-e^{-x})^2 + g'(e^{-x})(e^{-x}) =$$

$$= e^{-x}\left(g'(e^{-x}) - g''(e^{-x}) \cdot e^{-x}\right).$$

(C) Derivatives of order n

For the calculation of the derivative $f^{(n)}(x)$ we can use one of the following methods:

1. We calculate a few derivatives $(f', f'', f''', \dots)$ in order to deduce the expression of $f^{(n)}$, which we then demonstrate by way of induction.

2. We use Leibniz formula for the derivation of the product of two functions:

$$\left(f(x) \cdot g(x) \right)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(n-k)}(x) \cdot g(x)^{(k)}$$

The formula can be applied to a random quotient, because:

$$\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$$

(D) The study of derivability

In order to study the derivability of a function:

(a) we distinguish two categories of points that the domain has: points in which we know that the function is differentiable (being expressed by differentiable functions) and points in which we pursue the study of derivability (generally connection points between branches).

(b) for the study of derivability in this second category of points, we use one of the following methods:

(1) we calculate the lateral derivatives using the definition

(2) we calculate the lateral derivatives using the Corollary of Lagrange's theorem: If f is differentiable in a vicinity of x_0 and if it continues in x_0 and if there exists:

$$\lambda_s = \lim_{x \to x_0} f'(x)$$

$$(\lambda_d = \lim_{x \to x_0} f'(x)\text{, respectively})$$

then, f has a derivative on the left (on the right, respectively) in x_0 and:

$$f'_s(x_0) = \lambda_s$$

$$(f'_d(x_0) = \lambda_d\text{, respectively})$$

The continuity condition in x_0 that is required in the hypothesis of the corollary is essential for its application, but it doesn't constitute a restriction in the study of derivability, because if f isn't continuous in x_0, it isn't differentiable either.

EXAMPLE: Let's determine the parameters $\alpha, \beta \in \mathbb{R}$ so that the function:

$$f(x) = \begin{cases} \ln^2 x & x > e \\ \alpha x + \beta & x < e \end{cases}$$

is differentiable. What is the geometric interpretation of the result?

Answer: Method 1. (using the lateral derivatives)

a) in any point $x \neq e$, the function is differentiable, being expressed through differentiable functions.

b) we study the derivability in $x = e$.

Because of the proposition: ***f*** **non continuous ==>** ***f*** **non** differentiable, we study the continuity first.

$$l_s(e) = \lim_{\substack{x \to e \\ x < e}} f(x) = \lim_{\substack{x \to e \\ x < e}} (\alpha x + \beta) = \alpha e + \beta$$

$$l_d(e) = \lim_{\substack{x \to e \\ x < e}} f(x) = \lim_{\substack{x \to e \\ x < e}} \ln^2 x = 1$$

$$\text{and} \quad f(e) = \ln^2 e = 1$$

so: f continuous in $e <=> \alpha e + \beta = 1$. (4.3)

If the continuity condition in $x = e$ isn't met, the function is also not differentiable in this point, so **we study the derivability assuming the continuity condition is fulfilled**.

$$f'_s(e) = \lim_{\substack{x \to e \\ x < e}} \frac{f(x) - f(e)}{x - e} =$$

$$= \lim_{\substack{x \to e \\ x < e}} \frac{\alpha x + \beta - 1}{x - e} = [\, 1 = \alpha e + \beta \,] =$$

$$= \lim_{\substack{x \to e \\ x < e}} \frac{\alpha x + \beta - \alpha e - \beta}{x - e} = \alpha .$$

$$f'_d(e) = \lim_{\substack{x \to e \\ x < e}} \frac{f(x) - f(e)}{x - e} = \lim_{\substack{x \to e \\ x < e}} \frac{\ln^2 x - 1}{x - e}$$

For the calculation of this limit we can:

(1) apply the definition of the derivative,

(2) permute the limit with the logarithm,

(3) use l'Hospital's rule.

(1) We observe that by noting $g(x) = ln^2 x$, we have $g(e) = 1$ and the limit becomes:

$$\lim \frac{g(x) - g(e)}{x - e} = g'(e)$$

(because g is differentiable in e)

$$g'(x) = (\ln^2 x) = 2 \cdot \ln x \cdot \frac{1}{x} => g'(e) = \frac{2}{e},$$

$$\text{so } f'_d(e) = \frac{2}{e}$$

$$(2)\ \lim_{\substack{x \to e \\ x < e}} \frac{\ln^2 x - 1}{x - e} = \lim_{\substack{x \to e \\ x < e}} \frac{\ln^2 x - \ln^2 e}{x - e} =$$

$$= \lim_{\substack{x \to e \\ x < e}} \frac{(\ln x - \ln e)(\ln x + \ln e)}{x - e} =$$

$$= 2 \cdot \lim_{\substack{x \to e \\ x < e}} \frac{\ln x - \ln e}{x - e} = 2 \cdot \lim_{\substack{x \to e \\ x < e}} \frac{\ln(x/e)}{x - e} =$$

$$= \lim_{\substack{x \to e \\ x < e}} \left[\frac{1}{x - e} \cdot \ln\left(\frac{x}{e}\right)\right] = \lim_{\substack{x \to e \\ x < e}} \ln\left(\frac{x}{e}\right)^{\frac{1}{x-e}} =$$

$$= \ln \lim_{\substack{x \to e \\ x < e}} \left(\frac{x}{e}\right)^{\frac{1}{x-e}} = 1^{\infty}, \text{ etc.}$$

Therefore, f is differentiable in:

$$e \Leftrightarrow f'_s(e) = f'_d(e) \Leftrightarrow \alpha = \frac{2}{e}$$

and from the continuity condition we obtain $\beta = -1$. We thus have:

$$f(x) = \begin{cases} \ln^2 x & x > e \\ \frac{2}{e}x - 1 & x < e \end{cases}$$

The geometric interpretation of the result: the straight line $y = \frac{2}{e}x - 1$ is tangent to the curve $y = \ln x^2$.

If the branch of a function is expressed through a straight line $(y = \alpha x + \beta)$, the function is differentiable in the point x_0 of connection between the branches, if and only if the respective straight line is tangent in the point of abscissa x_0 to the graphic of the other branch.

Method 2: (Using the corollary of Lagrange's theorem)

(a) In any point $x \neq e$, the function is differentiable, being expressed by differentiable function and we have:

$$f'(x) = \begin{cases} \frac{2}{x} \ln x & x > e \\ \alpha & x < e \end{cases}$$

(b) We study the derivability in $x = e$. We first state the continuity condition:

$$l_s(e) = l_d(e) = f(e) \iff \alpha e + \beta = 1 .$$

In order to use the corollary of Lagrange's theorem we calculate the lateral limits of the derivative:

$$\lim_{\substack{x \to e \\ x < e}} \alpha = \alpha .$$

From the corollary we deduce $f'(e) = \alpha$. Analogously:

$$f'_d(e) = \lim_{\substack{x \to e \\ x < e}} f'(x) = \frac{2}{e} .$$

If f is differentiable in $x = e$ if and only if $\alpha = \frac{2}{e}$. From the continuity condition we deduce $\beta = -1$.

(E) Applications of the derivative in economics

(See the *Calculus manual, IXth grade*)

1. Let $\beta(x)$ be the benefit obtained for an expenditure of x lei. For any additional expenditure of h lei, the supplementary benefit on any spent leu is:

$$\frac{\beta(x + h) - \beta(x)}{h}$$

If h is sufficiently small, this relation gives an indication of the variation of the benefit corresponding to the sum of x lei.

If this limit exists:

$$\lim_{h \to 0} \frac{\beta(x+h) - \beta(x)}{h} = \beta'(x)$$

it is called the **bound benefit corresponding to the sum of x lei**.

2. Let $\gamma(p)$ be the total cost for the production of p units of a particular product. Then, the cost per each supplementary unit of product is:

$$\frac{\gamma(p+h) - \gamma(p)}{h}$$

and the limit of this relation, when h tends to zero, if it exists, is called **a bound cost** of production for p units of the considered product.

EXAMPLE: The benefit obtained for an expenditure of x lei is:

$$\beta(x) = x^2 - 3x + 2 .$$

For any additional expenditure of h lei, calculate the supplementary benefit per spent leu and the bound benefit corresponding to the sum of 1000 lei.

Answer: The supplementary benefit per spent leu is:

$$\frac{\beta(x+h) - B(x)}{h} = \frac{(x+h)^2 - 3(x+h) + 2}{h}$$

For $x = 1000$ this is equal to $1997 + h$ and

$$\lim_{h \to 0} \frac{\beta(x+h) - \beta(x)}{h} = 1997$$

Exercises

I. The following are required:

1. The equation of the tangent to the graphic $f(x) = \ln\sqrt{1+x^2}$ in the abscissa point $x_0 = 1$.

2. The equation of the tangent to the curve $f(x) = \sqrt{x^2 - k^2}$, that is parallel to $0x$.

3. The equation of the tangent to the curve $f(x) = x^3$, parallel to the first bisector.

4. The equation of the tangent to the curve $y = \frac{3x+2}{2x+6}$, that is parallel to the chord that unites the abscissae points $x = 1$ and $x = 3$.

5. Determine α and β so that $y = \alpha x + \beta$ and $y = \frac{x-1}{x}$ are tangent in $x = 1$. Write down their common tangent.

6. Show that the straight line $y = 7x - 2$ is tangent to the curve $y = x^2 + 4x$. *(Manual)*

II. 1. Calculate the derivative of the function:

$$f(x) = \int_0^{x^2} e^{t^6} \cdot \sin t^2 \, dt$$

2. Let $f: (-\infty, 0) \to \mathbb{R}$, $f(x) = x^2 - 3x$. Determine a subinterval $J \subseteq \mathbb{R}$, so that $f: (-\infty, 0) \to J$ is bijective. Let g be its inverse. Calculate $g'(-1)$ and $g''(-1)$. (Manual)

3. If:

$$|f(x) - f(y)| \leq M \cdot |x - y|^{1+\alpha},$$

with $\alpha > 0$ for any $x, y \in I$, the function f is constant on I.

4. If f has a limit in point a, then the function:

$$g(x) = (x - a) \cdot f(x)$$

is differentiable in a.

5. If f is bounded in a vicinity of x_0, then $g(x) = (x - x_0)^2 f(x)$ is differentiable in x_0. Particular case:

$$f(x) = \sin\left(\frac{1}{x}\right).$$

Indications:

1. Let f be a primitive of the function:

$$f(t) = e^{t^6} \sin t^2.$$

We have:

$$I(x) = F(x^4) - F(0), \text{ so } I'(x) = \left(F(x^4)\right).$$

3. For $x \neq y$ the inequality from the enunciation is equivalent to:

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M \cdot |x - y|^{\alpha}$$

from where, for $y \longrightarrow x$ we obtain $f(x) = 0$, so f is constant on I.

III. Calculate the derivatives of order n for:

1. $f(x) = \dfrac{1}{x - 3}$

2. $f(x) = \dfrac{1}{ax + b}$

3. $f(x) = \dfrac{1}{x^2 - 3x + 2}$

4. $f(x) = \ln(2x + 5)$

5. $f(x) = \ln(x^2 - 3x + 2)$

6. $f(x) = (x^3 - 2x^2 + 5x - 3) \cdot e^{x+1}$

7. $f(x) = \dfrac{\sqrt{x}}{x + 3}$

8. $f(x) = \dfrac{x^3}{(x - 2)(x - 3)}$

9. $f(x) = \text{arctg } x$

10. $f(x) = e^{ax} + e^{bx}$ from the expression of the derivative of order n, deduce Newton's binomial formula.

IV. 1. Show that:

a) $(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right)$,

b) $(\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)$.

2. Applying Leibniz's formula for:

$$f(x) = \frac{1}{x^2 - 3x + 2},$$

show that:

$$n! \cdot \left[\frac{1}{(x-1)^{n+1}} - \frac{1}{(x-2)^{n+1}}\right] =$$

$$= \sum_{k=0}^{n} C_n^k (n-k)! \cdot k! \cdot \frac{1}{(x-1)^{n+1} \cdot (x-2)^{n+1}}$$

Find similar formulas, using the functions:

$$f(x) = \frac{1}{x^2 - 5x + 6} \text{ and } f(x) = \frac{2x+1}{x^2 - 5x + 4}$$

3. Let $I = (0,1)$ and the functions $u, v: I \longrightarrow \mathbb{R}$, $u(x) =$ $u(x) = \inf_{y \in I}(x-y)^2$, $v(x) == \sup_{y \in I}(x-y)^2$. Study the derivability of the functions u and v and calculate $\sup_{x \in I} u(x)$, $\inf_{x \in I} v(x)$. *(Manual)*

4. If $f_n(x)$ is a sequence of differentiable function, having a limit in any point x, then:

$$\left(\lim_{n \to \infty} f_n(x)\right)' = \lim_{n \to \infty} f'(x) \qquad \text{(Manual)}$$

V. Study the derivability of the functions:

$$1.\ f(x) = \begin{cases} \ln(x^2 + 3x) & x \in (0,1) \\ \frac{5}{4}(x-1) + 2 \cdot \ln 2 & x \geq 1 \end{cases}$$

2. $f(x) = \min(x^2 - x,\ 4x - 2)$

3. $f_1(x) = |(x - 2)(x - 3)|$

$f_2(x) = |(x - 2)^2 \cdot (x - 3)|$

$f_3(x) = |(x - 2)^3 \cdot (x - 3)|$

$$4.\ f(x) = \begin{cases} \dfrac{-\pi}{4} & x = -1 \\ \displaystyle\int_0^x \frac{t^2}{\sqrt{1 - t^2}}\,dt & x \in (-1, 1) \\ \dfrac{\pi}{4} & x = 1 \end{cases}$$

5. $f'(x)$ **if** $f(x) = \begin{cases} \ln^2(1 - x) & x \leq 0 \\ tg^2 x & x \geq 0 \end{cases}$

$$6.\ f(x) = \begin{cases} \dfrac{1}{2}\left(1 + \dfrac{|x|}{x}\right) + \dfrac{1}{1 + e^{1/x}} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

$$7.\ f(x) = \begin{cases} \dfrac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

is indefinitely differentiable on ℝ and:

$$f^{(n)}(x) = \left(\frac{1}{x^{n+1}}\right) \cdot \int_0^x t^n \cdot \cos\left(t + \frac{n\pi}{2}\right) dt$$

8. $$f(x) = \begin{cases} \dfrac{x}{x-1} & x \le 0 \\ x \cdot \ln x & x \in (0,1) \\ \dfrac{e^{-x}}{e} & x \ge 1 \end{cases}$$

9. $$f(x) = \begin{cases} \arccos(\cos x) & x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \\ \arcsin(\sin x) & x \in \left(\dfrac{\pi}{2}, 2\pi\right] \end{cases}$$

10. $$f(x) = \frac{|x-1| \cdot e^{nx} + a(x+1)^2 \cdot e^{-nx}}{e^{nx} + e^{-nx}},$$

$a \in \mathbb{R}$.

V. Fermat, Rolle, Lagrange, Cauchy Theorems

(A) Fermat's theorem

1. Enunciation

If $f: [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then in any point of extremum from (a, b) (from the interior of the interval), the derivative is annulled. (Fermat, 1601-1665)

Observation: $x_0 \in (a, b)$ point of extremum $==> f'(x_0) = 0$. If x_0 is a point of extremum at the ends of the interval, it is possible that the derivative isn't annulled in x_0.

Example:

$f(x) = 3x + 2$, $f:[-1,2] \longrightarrow \mathbb{R}$

has two points of extremum: in $x_1 = -1$ and in $x_2 = 2$, and:

$f'(-1) = f'(2) = 3$.

2. Geometric and algebraic interpretation

a) The geometric interpretation results from the geometrical interpretation of the derivative: if the conditions of Fermat's theorem are fulfilled in any point from the interior of the interval, the tangent to the function's graphic is parallel to the axis $0x$.

b) The algebraic interpretation: if the conditions of Fermat's theorem are fulfilled on $[a, b]$, any point of extremum from (a, b) is a root to the equation $f'(x) = 0$.

Exercises

1. If $a_1, a_2, \dots a_n$ are positive numbers, so that:

$$a_1^x + a_2^x + \dots + a_n^x \geq n$$

for any $x \in \mathbb{R}$, then we have:

$$a_1 \cdot a_2 \cdot \dots \cdot a_n = 1$$

(Manual)

2. Let $a_1, a_2, \dots a_n, b_1, b_2, \dots b_n$ be positive real numbers, so that:

$$a_1 \cdot b_1^x + a_2 \cdot b_2^x + \dots + a_n \cdot b_n^x \geq a_1 + a_2 + \dots + a_n$$

for any $x \in \mathbb{R}$. Then :

$$b_1^{a_1} \cdot b_2^{a_2} \cdot \dots \cdot b_n^{a_n} = 1$$

(generalization of the previous exercise)

3. If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b) = 0$, then:

(i) if f' is increasing, it follows that $f(x) \leq 0$ on $[a, b]$;

(ii)) if f'' is decreasing, it follows that $f(x) \geq 0$ on $[a, b]$.

4. If $a^x \geq x^a$ for any $x > 0$, then $a = e$.

5. $\frac{tg\, a}{tg\, b} < \frac{a}{b}$ if $0 < a < b < \frac{\pi}{2}$.

SOLUTIONS:

1. The exercise is a particular case of exercise 2.

2. We highlight a function that has a point of extremum (global) on $\mathbb{R}$, observing that the inequality from the hypothesis can be written:

$f(x) \geq f(0)$, **with:**

$$f(x) = a_1 \cdot b_1^x + a_2 \cdot b_2^x + \dots + a_n \cdot b_n^x .$$

Because this inequality is true for any real x, it follows that $x = 0$ is a global point of minimum for f. According to Fermat's theorem, in this point the derivative is annulled, so $f'(0) = 0$. We have:

$$f'(x) = a_1 \cdot b_1^x \cdot \ln b_1 + a_2 \cdot b_2^x \cdot \ln b_2 + \dots +$$
$$+ a_n \cdot b_n^x \cdot \ln b_n$$

so: $f'(0) = 0 <=> a_1 \cdot \ln b_1 + a_2 \cdot \ln b_2 +$

$$+ \dots a_n \cdot \ln b_n = 0 <=> b_1^{a_1} + b_2^{a_2} + \dots + b_n^{a_n} = 1$$

3. (i) Let's assume there exists $c \in (a, b)$ so that $f(c) > 0$. We can even assume that c is a point of maximum, because such a point exists, according to Rolle's theorem, so we have $f'(c) = 0$.

Between c and b at least one point d exists in which $f'(d) < 0$, because, if, let's say $f'(x) \geq 0$ for any $x \in (c, b)$, it follows that f is increasing on (c, b) and so $f(c) < f(b) = 0$ implies $f(c) = 0$. Then, from $f'(d) =\leq 0 = f'(c)$ we deduce the contradiction:

$$f'(d) < f'(c) \text{ with } c < d$$

4. $a^x \geq x^a <=> x. \ln a \geq a. \ln x <=> \frac{\ln a}{a} \geq \frac{\ln x}{x}$ (for any $x > 0$). So a is the abscissa of the maximum of the function:

$$f(x) = \frac{\ln x}{x} .$$

We have:

$$f'(x) = 0 <=> x = e \text{ , so } a = e .$$

5. $\frac{tg\ a}{tg\ b} < \frac{a}{b} <=> \frac{tg\ a}{a} < \frac{tg\ b}{b}$

(if $0 < a < b < \frac{\pi}{2}$).

It is sufficient to prove that $f(x) = \frac{tg\, x}{x}$ is increasing, i.e. $f'(x) > 0$.

(B) Rolle's theorem

1. Enunciation

> If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$, then there exists $c \in (a, b)$ so that $f'(c) = 0$. (Rolle, 1652-1719)

Fermat's theorem states that in a point of extremum from the interior of an interval the derivative is annulled, but it doesn't mention when such a point exists.

Rolle's theorem provides a **sufficient condition** for the existence of at least one such point, adding to the hypothesis from Fermat's theorem the condition: $f(a) = f(b)$.

2. Geometric and algebraic interpretation

a) The geometric interpretation: if the conditions of Rolle's theorem are met, **there exists at least one point** in the interval (a, b)in which the tangent to the graphic is parallel to $0x$.

b) The algebraic interpretation: if the conditions of Rolle's theorem are met, **the equation $f'(x) = 0$ has at least one root** in the interval (a, b).

The algebraic interpretation of the theorem highlights a method used to prove that the equation $f(x) = 0$ has at least **one root** in the interval (a, b). For this, it is sufficient to consider a primitive F of f, for which the conditions of Rolle's theorem are fulfilled on $[a, b]$. It results that the equation $F'(x) = 0$ has at least one root in the interval (a, b), i.e. $f(x)$ has a root in the interval (a, b).

A second method to show that the equation $f(x) = 0$ has at least one root in the interval (a, b) is to show just that f is continuous and $f(a). f(b) < 0$.

3. Consequences

1. Between two roots of a differentiable function on an interval, there exists at least one root of the derivative.

2. Between two roots of a function's derivative on an interval, there exists, at most a root of the function.

Consequence 2. allows us to determine the number of roots of a function on an interval with the help of its derivative's roots (Rolle's sequence):

Let $x_1, x_2, \dots, x_n$ be the derivative's roots.

Then f has as many real, simple roots as there are variations of sign in the sequence:

$$f(-\infty),\ f(x_1),\ f(x_2),\ \dots,\ f(x_n),\ f(\infty)$$

(if we have $f(x_i) = 0$, then x_i is a multiple root)

Exercises

I. Study the applicability of Rolle's Theorem for the functions:

$$1.\ f(x) = \begin{cases} x^2 - 8x + 1 & x \in [2,4] \\ -15 & x \in (4,5) \\ x^2 - 10x + 10 & x \in [5,7] \end{cases}$$

$$2.\ f(x) = \begin{cases} x^2 - 2x + 2 & x \in [1,2] \\ 1 & x \in (-1,1) \\ x^2 + 2x + 2 & x \in [-2,-1] \end{cases}$$

$$3.\ f(x) = \begin{cases} \sin x & x \in \left[0, 2\pi\right] \\ 1 & x \in \left(\frac{\pi}{2}, 2\pi\right) \\ \cos x & x \in \left[2\pi, \frac{9\pi}{2}\right] \end{cases}$$

What is the geometric interpretation of the result?

Indication: 1. We obtain $c \in [4,5]$. In any point from the interval $[4,5]$, the graphic coincides with the tangent to the graphic.

II. Given:

$$a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathbb{R}.$$

Show that the equation:

$$\sum_{k=1}^{n} (a_k \cdot \cos kx + b_k \cdot \sin kx) = 0$$

has at least one solution in the interval $(0, 2\pi)$. (Manual)

2. If:

$$\sum_{i=0}^{n} \frac{i}{i+1} = 0,$$

then the equation:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots. a_1 x + a_0 = 0$$

has at least one root in the interval $(0,1)$.

3. (i) If:

$$a_1, a_2, \ldots a_n \neq 0,$$

then the equation:

$$n \cdot a_n \cdot x^{n-1} + (n-1) \cdot a_{n-1} x^{n-2} + \ldots. + a_1 = 0$$

has at least one root in the interval $(0,1)$.

(ii) in what conditions the same equation has at least one root in the interval $(-1,0)$?

(iii) The same question for the equations:

$$a_{2n} \cdot x^{2n} + a_{2n-1} \cdot x^{2n-1} + \ldots. + a_0 = 0$$

$$a_{2n+1} \cdot x^{2n+1} + a_{2n} \cdot x^{2n} + \ldots. + a_0 = 0$$

on the interval $(-1,1)$.

4. Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable and $a_1 < a_2 < \cdots < a_n$, roots of f. Show that f'has at least $n-1$ roots.

(Manual)

Consequences:

a) A polynomial function of degree n has, at most, n real, distinct zeros.

b) If all the roots of a polynomial are **real and distinct,** its derivative has the same property.

c) If all the roots of a polynomial are **real,** then its derivative has the same property.

5. Given:

$$f(x) = (x^2 - 1)^n.$$

Show that the equation $f^n(x) = 0$ has n distinct roots in the interval $(-1,1)$.

6. If f is n times differentiable on I and it has $n + 1$ distinct roots on I, then $f^n(x)$ has at least one root on I.

7. Let $f, g: [a, b] \to \mathbb{R}$, continuous on $[a, b]$, differentiable on (a, b) with $g(x) \neq 0$ and $g'(x) \neq 0$ on [a,b], and $\frac{f(a)}{g(a)} = \frac{f(b)}{g(b)}$.

Show that $c \in (a, b)$ exists, so that:

$$\frac{f(c)}{g(c)} = \frac{f'(c)}{g'(c)}$$

8. Let$f: [a, b] \to \mathbb{R}$, continuous on (a, b), differentiable on (a, b). Then between two roots of f there exists at least a root of $\alpha . f + f'$.

9. If the differentiable functions f and g have the property:

$$f'(x) \cdot g(x) - f(x) \cdot g'(x) = 0$$

on an interval, then between two of its roots there is a root of g and vice versa.

Consequence: If $f'(x). \cos x + f(x). \sin x \neq 0$, in any length interval bigger than π there is at least one root of f.

10. If $f, g: [a, b] \to \mathbb{R}$, continuous on $[a, b]$, differentiable on (a, b) , $g(x) \neq 0$, $g'(x) \neq 0$ and $f(a) = f(b) = 0$, then $c_n \in (a, b)$ so that:

$$\frac{f(c_n)}{g(c_n)} = \frac{1}{n} \cdot \frac{f'(c_n)}{g'(c_n)}$$

SOLUTIONS:

7. The required equality can also been written:

$$f(c) \cdot g'(c) - f'(c) \cdot g(c) = 0$$

This equality appears in points c that are roots of the derivative of $h(x) = \frac{f(x)}{(x)g}$, so it is sufficient to demonstrate that the function h fulfills the conditions of Rolle's theorem on $[a, b]$.

8. The equation $\alpha . f(x) + f'(x) = 0$ comes from the equalizing with zero the derivative of the function $F(x) = e^{\alpha x} f(x)$, so it is sufficient to show that F satisfies the conditions of Rolle's theorem.

9. Let x_1, x_2 be roots of f. By hypothesis, we have$g(x_1) \neq 0$ and $g(x_2) \neq 0$. If, for example, there can be found a root of g between x_1, x_2, this means that the function $h(x) = \frac{f(x)}{(x)g}$ satisfies the conditions of Rolle's theorem and so $h'(x)$ is annulled in at least one point between x_1 and x_2, which contradicts the hypothesis.

10. $\exists\, c_n \in (a,b)\ \frac{f(c_n)}{g(c_n)} = \frac{1}{n} \cdot \frac{f'(c_n)}{g'(c_n)} <=>$

$<=> \exists\, c_n \in (a,b)\ \ n \cdot f(c_n) \cdot g'(c_n) -$

$- g(c_n) \cdot f'(c_n) = 0$ <=> **the equation:**

$$n \cdot f(x) \cdot g'(x) - g(x) \cdot f'(x) = 0$$

has at least one root in $(a, b) <=> h'(x) = 0$ has at least one root in (a, b), where $h(x) = \frac{f(x)}{g^n(x)}$.

We will present next, two methods for the study of an equation's roots. The first of these uses Rolle's Theorem and the other is a consequence of the fact that any continuous function has Darboux's property.

Methods for the study of an equation's roots

1. Using Rolles's theorem (its algebraic interpretation)

2. Using Darboux's properties

(if f has Darboux's property (particularly if it is continuous) on $[a, b]$ and: $f(a).f(b) \leq 0$, then f has a root in $[a, b]$).

Examples:

1) If $f: [a, b] \to [a, b]$ is a continuous function, then $u, v \in [a, b]$ exists so that $f(u) = u$ and $f(v) = v$. *(Manual)*

2) Let $f: [0, 2\pi] \to \mathbb{R}$ be a continuous function so that $f(0) = f(2\pi)$. Show that $c \in [0, \pi]$ exists so that $f(c) = f(c + n)$. *(Manual)*

Consequences: If a traveler leaves in the morning from spot $A$and arrives, in the evening in spot B, and the next day he leaves and reaches again spot A, show that there is a point between A and B where the traveler has been, **at the same hour**, in both days.

Indication: If $S(t)$ is the space covered by the traveler, we have: $S(0) = S(24)$ and so $t_0 \in [0,12]$ exists, so that: $S(t_0) = S(t_0 + 12)$.

By formulating the problem like this, the result might seem surprising, but it is equal to saying that two travelers that leave, one from spot A heading to spot B, and the other from B to A, meet on the way.

3. Using Rolle's sequence

Example: Let's study the nature of the equation's roots:

$$x^3 - 3x^2 + a = 0$$

a being a real parameter.

Answer: The derivative's roots are:

$$x' = 0 \quad \text{and} \quad x'' = 2$$

and we have:

$$f(x') = a \quad \text{and} \quad f(x'') = a - 4 .$$

The results are illustrated in the following table:

x	$-\infty$	0	2	$+\infty$	
$f(x)$	$-\infty$	a	$a-4$	$+\infty$	**The nature of the roots**
$a<0$	$-$	$-$	$-$	$+$	a real root $x_1 > 2$
$a=0$	$-$	0	$-$	$+$	$x_1 = x_2 = 0,\ x_3 > 2$
$a \in (0,4)$	$-$	$+$	$-$	$+$	$x_1 \in (-\infty, 0),\ x_2 \in (0,2),\ x_3 > 2$
$a=4$	$-$	$+$	0	$+$	$x_1 < 0,\ x_2 = x_3 = 2$
$a>4$	$-$	$+$	$+$	$+$	a real root $x_1 < 0$

4. The graphical method.

The equation $F(x,m) = 0$ is reduced to the form $f(x) = a$. The number of roots is equal to the number of intersection points between the graphics: $y = f(x)$ and $y = m$.

Example:

$$x^3 + mx^2 + 2mx + 1 = 0 \iff m = -\frac{x^3 + 1}{x(x+2)} .$$

The number of roots of the given equation is equal to the number of intersection points between the graphics:

$$y = -\frac{x^3 + 1}{x(x+2)} \quad \text{and} \quad y = m$$

5. Viette's relations

Example: Show that the equation:

$$x^4 + (a+1)\cdot x^3 + \left(a^2 + \frac{a}{2} + 1\right)\cdot x^2 + b\cdot x + c = 0,$$

with $a, b, c \in \mathbb{R}$ has, at most, two real roots.

Answer: we have:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = -a - 1 \\ x_1x_2 + x_2x_3 + \ldots + x_3x_4 = a^2 + \dfrac{a}{2} + 1 \end{cases} \Rightarrow$$

$$\Rightarrow (x_1 + x_2 + x_3 + x_4)^2 = a^2 + 2a + 1 \Leftrightarrow$$

$$\Leftrightarrow x_1^2 + x_2^2 + x_3^2 + x_4^2 +$$

$$+ 2\cdot(x_1x_2 + x_2x_3 + \ldots + x_3x_4) = a^2 + 2a + 1$$

$$\Leftrightarrow x_1^2 + x_2^2 + x_3^2 + x_4^2 = -a^2 + a - 1 < 0,$$

so the equation has at least two complex roots.

6. Using the theorem of the average

Theorem: If $f: [a, b] \to \mathbb{R}$ is a Riemann integral (so it is bounded), $\mu \in [m, M]$ exists, so that:

$$\int_a^b f(x)\,dx = \mu(b - a)$$

Example: Let $f: [0,1] \to \mathbb{R}$ be continuous, so that:

$$2\int_0^1 f(x)\,dx = 1.$$

Show that the equation $f(x) = x$ has a root $x_0 \in (0,1)$.

Answer: $f(x) = x <=> f(x) - x = 0$, so by noting: $h(x) = f(x) - x$ we have to show that the equation $h(x) = 0$ has a root in the interval $(0,1)$. We have:

$$\int_0^1 f(x)\,dx = 1, \Longleftrightarrow 2\int_0^1 (h(x)+x)\,dx = 1 \Longleftrightarrow$$

$$\Longleftrightarrow 2\int_0^1 h(x) = 0.$$

But: $f_0^2\, h(x) = 0$, according to the theorem of the average, with $\mu \in (m, M)$. The function f, being continuous, has Darboux's property, so, for μ there exists $x_0 \in (0,1)$so that $\mu = h(x_0)$.

Exercises

1. The equations:

a) $x^4 - x^3 + x^2 + \alpha x + \beta = 0$

b) $x^4 - (\sin \alpha) \cdot x^3 + x^2 + \beta x + \gamma = 0$

c) $x^4 - \sqrt{2\alpha}\, x^3 + \alpha x^2 + \beta x + 1 = 0$

can't all have real roots if $\alpha, \beta, \gamma \in \mathbb{R}$.

Indication:

a) $x_1^2 + x_2^2 + x_3^2 + x_4^2 < 0$

b) $x_1^2 + x_2^2 + x_3^2 + x_4^2 = \sin^2\alpha - 2 < 0$

c) $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$, $x_1 \cdot x_2 \cdot x_3 \cdot x_4 < 0$

2. If $f: [0,1] \to \mathbb{R}$ is continuous and with the property:

$$\int_0^1 f(x)\, dx = \frac{2}{\pi},$$

then, the equation $f(x) - \sin \pi x = 0$ has one root in $(0,1)$.

3. If $f: [0,1] \to \mathbb{R}$ is continuous and with the property:

$$6\int_0^1 f(x)\, dx = 2a + 3b + 6c,$$

then, the equation $f(x) - ax^2 - bx - c = 0$ has one root in $(0,1)$.

4. If $f: [0,1] \to \mathbb{R}$ is continuous and $n > 1$ exists for which:

$$\int_0^1 f(x)\, dx = 1 + \frac{1}{2} + \dots + \frac{1}{n},$$

the equation: $(1 - x)f(x) = 1 - x^n$ has one root in $(0,1)$.

Indication: We apply the theorem of the average to the function:

$$h(x) = f(x) - (1 + x + \dots + x^{n-1})$$

(C) Lagrange's Theorem

1. Enunciation:

Let $f:[a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and differentiable on (a,b). Then $c \in (a,b)$ exists, so that:

$$f(b) - f(a) = f'(c) \cdot (b - a)$$

(Lagrange, 1736 – 1813)

2. Geometric and algebraic interpretation

a) Geometric interpretation:

If the conditions of the theorem are met, there exists a point $c \in (a,b)$ in which the tangent to the graphic is parallel to the chord that unites the graphic's extremities.

b) Algebraic interpretation

If the conditions of the theorem are met, the equation:

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

has at least one root in the interval (a,b).

3. The corollary of Lagrange's Theorem

If f is continuous on an interval I and differentiable on $I \setminus \{x_0\}$ and in x_0 there exists the derivative's limit:

$$\lim_{x \to x_0} f'(x) = \lambda ,$$

then f is differentiable in x_0 and:

$$f'(x_0) = \lambda .$$

This corollary facilitates the study of a function's derivability in a point in an easier manner that by using the lateral derivatives.

Examples:

1. Study the derivability of the function:

$$f(x) = \begin{cases} 2^{\frac{1}{x-1}} & x < 1 \\ \ln(x^2 - 2x + 2) & x \geq 1 \end{cases}$$

Solution: In any point $x \neq 1$, the function is differentiable, being expressed by differentiable functions. We study the derivability in $x = 1$ using the corollary of Lagrange's theorem.

- the continuity in $x = 1$:

$$l_s(1) = \lim_{\substack{x \to 1 \\ x < 1}} f(x) = \lim_{\substack{x \to 1 \\ x < 1}} 2^{\frac{1}{x-1}} = 0$$

$$l_d(1) = \lim_{\substack{x \to 1 \\ x > 1}} f(x) = \lim_{\substack{x \to 1 \\ x > 1}} \ln(x^2 - 2x + 2) = 0$$

$f(1) = \ln 1 = 0$, so f is continuous on $\mathbb{R}$ (and differentiable on $\mathbb{R}\{1\}$).

- derivability. For the study of derivability in $x = 1$, we have:

$$f'(x) = \begin{cases} -\dfrac{1}{(x-1)^2} \cdot 2^{\frac{1}{x-1}} \cdot \ln 2 & x < 1 \\ \dfrac{2(x-1)}{x^2 - 2x + 2} & x > 1 \end{cases}$$

(we know that f' exists just on $\mathbb{R} - (1)$).

Exercises

I. Study the applicability of Lagrange's theorem and determine the intermediary points c for:

$$1.\ f(x) = \begin{cases} 2x - 2 & x \in [0,1) \\ x^2 + 1 & x \in [1,2] \\ 4x - 8 & x \in (2,3] \end{cases}$$

$$2.\ f(x) = \begin{cases} \sin x & x \in [0, \frac{\pi}{6}] \\ x^2 + \alpha \cdot x + \beta & x \in (\frac{\pi}{6}, \frac{\pi}{2}] \end{cases}$$

$$3.\ f(x) = \left| \left(x - \frac{\pi}{2}\right)^3 \cdot \cos x \right|,\ x \in [0, \pi]$$

$$4.\ f(x) = \operatorname{arctg} x + \operatorname{arctg} \frac{1-x}{1+x},\ x \in [0, n]$$

$$5.\ f(x) = \begin{cases} \left(e^{-\frac{1}{x^2}} \right)^{(n)} & x \in [-1, 1],\ x \neq 0 \\ 0 & x = 0 \end{cases}$$

Indication:

2. The continuity condition in $x = \frac{\pi}{6}$ is:

$$\frac{\pi^2}{36} + \alpha \frac{\pi}{6} + \beta = \frac{1}{2} ,$$

and the derivability condition (the corollary to Lagrange's theorem) is:

$$2 \frac{\pi}{6} + \alpha = \frac{\sqrt{3}}{2} .$$

Point c is the solution of the equation:

$$f'(x) = \frac{f\left(\frac{\pi}{2}\right) - f(0)}{\frac{\pi}{2} - 0} = 0 .$$

II. 1. Let $f(x) = ax^2 + bx + c$. Apply the theorem of the finite increases on the interval $[x_1, x_2]$, finding the intermediary point c. Deduce from this a way to build the tangent to the parabola, in one of its given points.

2. We have $m \leq f'(x) \leq M$ for any $x \in I$, if and only if:

$$m \cdot |x - y| \leq |f(x) - f(y)| \leq$$

$$\leq M \cdot |x - y| \quad \forall\ x, y \in I . \qquad \text{(L1)}$$

3. (The generalization of Rolle's theorem). If f is continuous on $[a,b]$and differentiable on (a,b), then $c \in (a,b)$ exists so that:

$$\operatorname{sgn}[f(b) - f(a)] = \operatorname{sgn} f'(c) \qquad (L2)$$

4. Supposing f is twice differentiable in a vicinity V of point a, show that for h, small enough, there exists $p, q \in V$ so that:

$$\text{a)} \quad \frac{f(a+h) - f(a-h)}{2h} = f'(p)$$

$$\text{b)} \quad \frac{f(a+h) - f(a-h) - 2f(a)}{h^2} = f''(q)$$

(Manual)

5. Let $f: [0,\infty) \to \mathbb{R}$ be a differentiable function so that:

$$\lim_{x \to \infty} f(x) = 0$$

and let $a > 0$ be fixed. Applying the Lagrange theorem on each interval:

$$[a+n, a+n+1], n \in \mathbb{N},$$

show there exists a sequence $(x_n)_{n\in\mathbb{N}}$, having the limit infinite and so that:

$$\lim_{x \to \infty} \sum_{i=0}^{n} f'(x_i) = -f(a)$$

6. The theorem of finite increases can also be written:

$$f(x+h) - f(x) = h.f'(x+ch) \text{ with } c \in (0,1).$$

Apply this formula to the functions:

$$\text{a)} \ f(x) = x^2 \quad \text{b)} \ f(x) = x^3,$$

$$\text{c)} \ f(x) = mx + n$$

and study the values corresponding to the real point c.

SOLUTIONS:

1. $c = \frac{x_1+x_2}{2}$, so, the abscissa point c being given, in order to build the tangent in point $(c, f(c))$ of the graphic, we consider two points, symmetrical to c:

$x_1 = c - \varepsilon$, $x_2 = c + \varepsilon$.

The tangent passes through $(c, f(c))$ and is parallel to the chord determined by the points:

$$(x_1, f(x_1)) \text{ and } (x_2, f(x_2)).$$

2. The necessity: For $x = y$ $(L1)$ is verified, and for $x \neq y$, we have:

$$(L1) \Leftrightarrow m \leq \left| \frac{f(x) - f(y)}{x - y} \right| \leq M \Leftrightarrow$$

$$\Leftrightarrow m \leq f'\left(c_{x_j}\right) \leq M$$

inequalities which are true due to the hypothesis.

Reciprocally, let $x \in I$, randomly. Making y tend to x, for:

$$x \neq y \quad (L1) \Leftrightarrow m \leq \left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

we have:

$$m \leq f'(x) \leq M .$$

3. We distinguish three cases:

(1) $f(a) < f(b)$

In this case we must prove that $c \in (a, b)$ exists, so that: $f'(c) > 0$. If we had, let's say, $f'(c) \leq 0$ on (a, b), it would follow that f is decreasing, so $f(a) > f(b)$.

(2) $f(a) = f(b)$

We are situated within the conditions of Rolle's theorem.

(3) The case $f(a) > f(b)$ is analogous to (1).

4. a) We apply Lagrange's theorem to f on $[a - h, a + h]$.

b) We apply Lagrange's theorem twice to the function $g(x) = f(a + x) - f(x)$ on the interval $[a - h, a]$.

6. a) $c = 1/2$, c) is obtained $mh = mh$, so c cannot be determined.

(D) Cauchy's Theorem

1.Enunciation

Let If $f, g: [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) so that:

$g'(x) \neq 0 \quad \forall x \in (a,b).$

Then $c \in (a, b)$ exists, so that:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

(Cauchy, 1789-1857)

2. Geometric and algebraic interpretation

a) The geometric interpretation: in the conditions of the theorem, there exists a point in which the relation of the tangents' slopes to the two graphics is equal to the relation of the chords' slopes that unite the extremities of the graphics.

b) The algebraic interpretation: in the conditions of the theorem, the equation:

$$f'(x)\big(g(b) - g(a)\big) - g'(x)\big(f(b) - f(a)\big) = 0$$

has at least one root in (a, b).

Exercises

I. Study the applicability of Cauchy's theorem and determine the intermediary point c for:

1. $f(x) = \ln x$, $g(x) = \frac{e}{x}$, $f,g:[1,e] \longrightarrow \mathbb{R}$

2. $f(x) = \ln x$, $g(x) = 2x - 1$, $f,g:[1,e] \longrightarrow \mathbb{R}$

3. $f(x) = \sin x$, $g(x) = \cos x$,

$f,g: \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ (Manual)

$$4.\ f(x) = \begin{cases} \dfrac{x^3}{3} - x^2 + 1 & x \in [1,3] \\ -x + \dfrac{4}{3} & x \in [0,1] \end{cases}$$

$$g(x) = x^2 + 1\ ,\ f,g:[0,3] \longrightarrow \mathbb{R}$$

II. If f has derivatives of any order on $[a, b]$, by applying Cauchy's formula to the functions:

a) $g(x) = f(x)$, $h(x) = b - x$, show that $c \in (a, b)$ exists, so that:

$$f(b) = f(a) + (b - a)f'(c)\ .$$

b) $g(x) = f(x) + (b - x)f'(x)$, $h(x) = (b - x)^2$,

show that $c \in (a, b)$ exists, so that:

$$f(b) = f(a) + (b - a)f'(a) + \frac{(b - a)^2}{2!}f''(c)$$

c) $g(x) = f(x) + \dfrac{b - x}{1!}f'(x) +$

$$+ \frac{(b - x)^2}{2!}f''(x),\quad h(x) = (b - x)^3,$$

show that $c \in (a, b)$ exists, so that:

$$f(b) = f(a) + \frac{(b - a)}{1!}f'(a) +$$

$$+ \frac{(b - a)^2}{2!}f''(a) + \frac{(b - a)^3}{3!}f'''(c)$$

d) $g(x) = f(a) + \dfrac{b - a}{1!}f'(x) + \dfrac{(b - x)^2}{2!} +$

$$+ \dots + \frac{(b - x)^n}{n!}f^{(n)}(x),\ g(x) = (b - x)^{n+1},$$

show that $c \in (a, b)$ exists, so that:

$$f(b) = f(a) + \frac{b - a}{1!}f'(a) + \frac{(b - a)^2}{2!}f''(a) +$$

$$+ \dots + \frac{(b - a)^n}{n!}f^{(n)}(a) + \frac{(b - a)^{n+1}}{(n + 1)!}f^{(n+1)}(c)$$

2. Applying the conclusion from point $b)$ show that there exists no functions $f: \mathbb{R} \to \mathbb{R}$ twice differentiable, so that:

$f(x) \geq 0$ and $f''(x) < 0$

(there are no non-negative and strictly concave functions on the entire real axis).

SOLUTION:

2. From $f''(x) < 0$ for any $x \in \mathbb{R}$, we deduce that f' is strictly increasing and so we cannot have $f'(x) = 0$ for any $x \in \mathbb{R}$. Therefore, for any $x_0 \in \mathbb{R}$ exists, so that $f'(x) \neq 0$.

But then, from the relation:

$$f(x) = f(x_o) + (x - x_o)f'(x_o) + \frac{(x - x_o)^2}{2} f''(c)$$

it follows that:

$$f(x) - f(x_o) < (x - x_o)f'(x_o) \qquad (C1)$$

We have two situations:

(1) if $f'(x_0) > 0$, making x tend to $-\infty$ in $(C1)$, we have:

$$\lim_{x \to -\infty} \left(f(x) - f(x_o) \right) =$$

$$= \lim_{x \to -\infty} (x - x_o)f'(x_o) = -\infty$$

consequently:

$$\lim_{x \to -\infty} f(x) = -\infty \text{ — contradiction.}$$

2. If $f'(x_0) < 0$, we obtain the same contradiction if we make x tend to $+\infty$.

VI. Equalities and Inequalities

Equalities

It is known that if the derivative of a function is equal to zero **on an interval**, the respective function is constant **on that interval.** This observation allows us to prove certain equalities of the form:

$f(x) = g(x) + C.$

or, in particular,

$f(x) = g(x)$ and $f(x) = C = constant.$

Indeed:

$$f(x) = g(x) + C \iff f(x) - g(x) = C$$

and in order to prove this equality it is sufficient to prove that:

$(f(x) - g(x))' = 0.$

Observation: If the derivative is equal to zero on a reunion of disjunctive intervals, the constant may vary from one interval to the next.

To determine the constant C, we can use two methods:

1. We calculate the expression $f(x) - g(x)$ in a conveniently chosen point from the considered interval.

2. It the previous method cannot be applied, we can calculate:

$$\lim_{x \to x_0} \left(f(x) - g(x) \right)$$

x_0 also being a conveniently chosen point (one end of the interval).

Example: Show that we have:

$$\operatorname{arctg} x + \operatorname{arctg} \frac{1-x}{1+x} = \begin{cases} -\frac{3\pi}{4} & x \in (-\infty, -1) \\ \frac{\pi}{4} & x \in (-1, \infty) \end{cases}$$

Solution: For:

$$h(x) = \text{arctg } x + \text{arctg } \frac{1-x}{1+x}$$

we have $h'(x) = 0$ for any $x \in \mathbb{R}(-1)$, domain that is a reunion of disjunctive intervals. We calculate the value of the constant on each interval.

(1) For $x > -1$, we choose the point $x = 0$ where we can easily make the calculations:

$$h(0) = \text{arctg}(0) + \text{arctg}(1) = \frac{\pi}{4}.$$

(2) For $x < -1$, we can't find a point in which to easily calculate the value of h, but we observe that we can calculate:

$$\lim_{\substack{x \to -1 \\ x < -1}} h(x) = -\frac{\pi}{4} - \frac{\pi}{2} = -\frac{3\pi}{4}.$$

Inequalities

Method 1. Using Lagrange's or Cauchy's theorem.

In some inequalities, we can highlight an expression of the form:

$$\frac{f(b) - f(a)}{b - a}$$

f and the points a, bbeing conveniently chosen. In this case, we can use Lagrange's theorem to prove the respective inequality, by replacing the expression:

$$\frac{f(b) - f(a)}{b - a}$$

from the inequality, with $f'(c)$. The new form of the inequality can be proven taking into account the fact that $a < c < b$ and the monotony of f'.

A similar method can be used if in the inequality an expression of the following form is highlighted:

$$\frac{f(b) - f(a)}{g(b) - g(a)}$$

using Cauchy's theorem.

EXAMPLES:

1. Using Lagrange's theorem, show that:

$$\frac{b-a}{b} < \ln\frac{b}{a} < \frac{b-a}{a},$$

if $0 < a < b$.

Solution: We go through these steps:

(a) We highlight an expression of the form:

$$\frac{f(b) - f(a)}{b - a}$$

observing that:

$$\ln\frac{b}{a} = \ln b - \ln a$$

and dividing by $b - a$. The inequality becomes:

$$\frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$$

(b) we apply Lagrange's theorem to the function $f(x) = \ln x$ on the interval $[a, b]$. $c \in (a, b)$ exists so that:

$$\frac{\ln b - \ln a}{b - a} = f'(c) = \frac{1}{c}$$

(c) the inequality becomes:

$$\frac{1}{b} < \frac{1}{c} < \frac{1}{a}$$

This new form of the inequality is proven considering the fact that $a < c < b$ and that $f'(x) = \frac{1}{x}$ is decreasing.

2. Using Lagrange's theorem, show that $e^x > 1 + x$ for any $x \neq 0$.

Solution:

To use Lagrange's theorem, we search for an interval $[a, b]$ and a function so that we can highlight the relation $\frac{f(b)-f(a)}{b-a}$ in our inequality. For this we observe that, if $x \neq 0$, we have two situations:

(1) $x > 0$. In this case we can consider the interval $[o, x]$.

a) We highlight, in the given inequality, an expression of the form: $\frac{f(x)-f(a)}{x-0}$, observing that, if $x > 0$, we have:

$$e^x > 1 + x \iff \frac{e^x - 1}{x} > 1 .$$

(b) We apply Lagrange's theorem to the function:

$$f(t) = e^t$$

on the interval $[0, x]$. $c \in (0, x)$ exists so that:

$$\frac{e^x - 1}{x} = f'(c) = e^c.$$

(c) The inequality becomes:

$$e^c > 1.$$

This form of the inequality is proven considering the fact that $0 < c < x$ and the derivative $f'(x) = e^x$is increasing (so $e^0 < e^c$).

(2) If $x < 0$ the demonstration is done analogously.

Method 2: The method of the minimum

This method is based on the observation that if x_0 is a global point of minimum (the smallest minimum) for a function h on a domain D and if $h(x_0) \geq 0$ (the smallest value of h is non-negative), then $h(x) \geq 0$ for any $x \in D$ (all the values of h are non-negative). We can demonstrate therefore inequalities of the form $f(x) \geq g(x)$, i.e. $f(x) - g(x) \geq 0$, in other words, of the form: $h(x) \geq 0$.

Example: Using the method of the minimum, show that $e^x > 1 + x$ for any $x \neq 0$.

Solution: In order to use this method, we proceed as follows:

(a) We write the inequality under the form: $h(x) \geq 0$:

$$e^x > 1 + x \iff e^x - x - 1 > 0$$

(b) Let $h(x) = e^x - 1 - x$. Using the variation table, we calculate the global minimum (the smallest minimum) of h:

x	$-\infty$							0						$+\infty$
$h'(x)$		$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$
$h(x)$		$\searrow$		$\searrow$		$\searrow$		0	$\nearrow$		$\nearrow$		$\nearrow$	

From the table it can be observed that h has a single minimum, in $x = 0$, so this is also the global minimum. Its value is $h(0) = 0$. So, for $x \neq 0$, we have $h(x) > 0$.

Method 3: (Inequalities for integrals, without the calculation of the integrals)

To prove an inequality of the form:

$$\alpha \leq \int_a^b f(x)\,dx \leq \beta$$

without calculating the integral, we use the observation that if m, or M respectively, are values of the global minimum and maximum of f on the interval $[a, b]$, we have:

$$m \leq f(x) \leq M \quad \text{for any} \quad x \in [a,b]$$

and so, by integrating, we obtain:

$$m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a)$$

so, in order to prove the required inequalities, the following inequalities (numerical) still have to be proven:

$$\alpha \le m(b-a) \quad \text{and} \quad M(b-a) \le \beta$$

Example: Show that:

$$2\sqrt{e} \le \int_0^1 e^{x^2} dx + \int_0^1 e^{1-x^2} dx \le 1 + e$$

Solution: (a) The inequality can be written under the form:

$$2\sqrt{e} \le \int_0^1 \left[e^{x^2} + e^{1-x^2} \right] dx \le 1 + e$$

(b) Using the variation table, we calculate the global minimum and maximum of the function $f(x) = e^{x^2} + e^{1-x^2}$ on the interval $[0,1]$:

x	0			$\frac{1}{\sqrt{2}}$			1
$f'(x)$	0	$-$	$-$ $-$	0	$+$	$+$	$+$
$f(x)$	$1+e$	$\searrow$	$\searrow$	$2\sqrt{e}$	$\nearrow$	$\nearrow$	$1+e$

We thus have $m = 2\sqrt{e}$ and $M = 1 + e$

(c) We integrate the inequalities $m \le f(x) \le M$ on the interval $[a, b]$ and we obtain:

$$\int_0^1 2\sqrt{e}\, dx \le \int_0^1 f(x)\, dx \le \int_0^1 (1+e)\, dx \Leftrightarrow$$

$$\Leftrightarrow 2\sqrt{e} \le \int_0^1 f(x)\, dx \le 1 + e.$$

Method 4: Using the inequality from the definition of convex (concave) functions

Let I be an interval on a real axis.

Definition: The function $f: I \to \mathbb{R}$ is convex on I if between any two points $x_1, x_2 \in I$, the graphic of f is situated underneath the chord that unites the abscissae points x_1 and x_2.

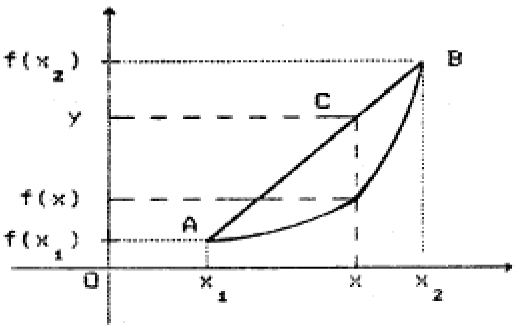


Fig. 6.1

With the notations from the adjacent graphic, this condition is expressed through the inequality: $f(x) \leq y$.

In order to explain x and y, we observe that:

1. x is in the interval $[x_1, x_2]$ if and only if it can be written in the form:

$$x = \alpha \cdot x_1 + (1 - \alpha) \cdot x_2, \text{ with } \alpha \in [0,1] \qquad (E.I.1)$$

Indeed, if $x \in [x_1, x_2]$, it is sufficient that we take $\alpha = (x - x_2)/x_1 - x_2)$ to meet the required equality.

Reciprocally, if x has the expression from $(E.I.1)$, in order to show that $x \in [x_1, x_2]$, the inequality system must be checked:

$$\begin{cases} \alpha \cdot x_1 + (1 - \alpha) \cdot x_2 \geq x_1 \\ \alpha \cdot x_1 + (1 - \alpha) \cdot x_2 \leq x_2 \end{cases}$$

The first inequality is equivalent to $\alpha \leq 1$, and the second inequality is equivalent to $\alpha \leq 0$.

Observation: The condition $\alpha \in [0,1]$ from $(E.I.1)$ is therefore essential.

The inequality from the convexity becomes:

$$f(\alpha \cdot x_1 + (1-\alpha) \cdot x_2) < y$$

2. To express y, we observe that:

$$\frac{x - x_1}{x_2 - x} = \frac{CA}{CB} = \frac{y - f(x_1)}{f(x_2) - y}$$

so:

$$\frac{\alpha \cdot x_1 + (1-\alpha) \cdot x_2 - x_1}{x_2 - \alpha \cdot x_1 - (1-\alpha) \cdot x_2} = \frac{y - f(x_1)}{f(x_2) - y}$$

from which it follows:

$$y = \alpha \cdot f(x_1) + (1-\alpha) \cdot f(x_2)$$

By replacing x and y, we obtain:

$f: I \to \mathbb{R}$ is convex <=>

$$\Leftrightarrow \left[\forall\ x_1, x_2 \in I\ \ \forall\ \alpha \in [0,1]\right.$$

$$\left. f(\alpha x_1 + (1-\alpha) x_2) < \alpha f(x_1) + (1-\alpha) f(x_2)\right]$$

$f: I \to \mathbb{R}$ is said to be concave if, for any $x_1, x_2 \in I$, between the abscissae points x_1 and x_2, the graphic of the function f is above the chord that unites these points, namely:

$$\forall\ x_1, x_2 \in I\ \ \forall\ \alpha \in [0,1]$$

$$f(\alpha x_1 + (1-\alpha) x_2) > \alpha f(x_1) + (1-\alpha) f(x_2)$$

It is known that we can verify the convexity and concavity of a function twice differentiable on an interval with the use of the second derivative:

$$f''(x) > 0 \text{ on } I \Leftrightarrow f \text{ convex on } I$$

$$f''(x) < 0 \text{ on } I \Leftrightarrow f \text{ concave on } I$$

Using the two ways of expressing convexity, we can formulate and prove certain inequalities.

Example: Show that for any x_1 and x_2 from $\mathbb{R}$, we have:

$$e^{\frac{x_1 + x_2}{2}} < \frac{e^{x_1} + e^{x_2}}{2}$$

Solution: The function $f(x) = e^x$ is convex on $\mathbb{R}(f''(x) > 0)$, so:

$$\forall\, x_1, x_2 \in \mathbb{R} \quad \forall\, \alpha \in (0,1)$$

$$f(\alpha x_1 + (1-\alpha)x_2) < \alpha f(x_1) + (1-\alpha)f(x_2)$$

from where, for $\alpha = \frac{1}{2}$ we obtain the inequality from the enunciation.

Exercises

I. Show that:

1. $\arcsin x + \arccos x = \frac{\pi}{2}$

for any $x \in [-1,1]$

2. $\operatorname{arcctg} x + \operatorname{arctg} \frac{1}{x} = \begin{cases} -\frac{\pi}{2} & x \in (-\infty, 0) \\ \frac{\pi}{2} & x \in (0, \infty) \end{cases}$

3. $2\operatorname{arctg} x + \arcsin \frac{2x}{1 + x^2} =$

$$= \begin{cases} \pi & x \in [1, \infty) \\ -\pi & x \in (-\infty, -1] \end{cases}$$

4. $\arcsin \frac{\cos\sqrt{x} + \sqrt{3}\cdot \sin\sqrt{x}}{2} -$

$- \sqrt{x} = \frac{\pi}{6}, \; x \in [0, 1 - \frac{\pi}{6}]$

5. For the intervals $I_1, I_2 \subset \mathbb{R}$ let's consider the continuous and injective functions:

$$f : I_1 \longrightarrow \mathbb{R}, \; g : I_2 \longrightarrow \mathbb{R}$$

and the constant c in the interior of $f(I_1)$. From the condition: $f\big(u(x)\big) = c + g(x)$ it follows that $u(x) = f^{-1}(c + g(x)$ for any x that fulfills the condition $c + g(x) \in f(I_1)$. Using this observation, construct equalities of the form $h(x) = c$ for:

(a) $f(x) = \arccos x, \quad g(x) = x^2$

(b) $f(x) = \operatorname{tg} x, \quad g(x) = 3x + 2$

(c) $f(x) = e^{x+1}, \quad g(x) = \operatorname{tg} x$

Answer: (a) We have:

$$f : [-1, 1] \longrightarrow \mathbb{R}, \; f([-1,1]) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

$$g : [0, \infty) \longrightarrow \mathbb{R}.$$

We determine function u from the condition $f\big(u(x)\big) = c + g(x)$ that becomes:

$\arccos u(x) = c + x^2$, so $u(x) = \cos(c + x^2)$.

For $c = \frac{\pi}{4}$, for example, we have:

$$u(x) = \frac{\sqrt{2}}{2}\left(\cos x^2 - \sin x^2\right),$$

and from the condition:

$$\frac{\pi}{4} + x^2 \in [-1, 1]$$

we deduce:

$$x \in \left[-\sqrt{1 - \frac{\pi}{4}}, \sqrt{1 - \frac{\pi}{4}} \right]$$

and so we have:

$$\arccos \frac{\sqrt{2}\,(\cos x^2 - \sin x^2)}{2} - x^2 = \frac{\pi}{4},$$

$$x \in \left[-\sqrt{1 - \frac{\pi}{4}}, \sqrt{1 - \frac{\pi}{4}} \right]$$

II. Using Lagrange's theorem, prove the inequalities:

1. $\dfrac{b - a}{\sin^2 b} < \operatorname{ctg} a - \operatorname{ctg} b < \dfrac{b - a}{\sin^2 a}$,

$[a,b] \subset \left(0, \dfrac{\pi}{2}\right)$

2. $\dfrac{b - a}{\sin^2 a} < \operatorname{ctg} a - \operatorname{ctg} b < \dfrac{b - a}{\sin^2 b}$,

$[a,b] \subset \left(\dfrac{\pi}{2}, \pi\right)$

3. $\dfrac{x + 1}{x + 2} < \ln(x + 2) < x + 1, \quad x > -1$

4. $e^{\frac{x_1 + x_2}{2}} < \dfrac{e^{x_1} + e^{x_2}}{2}, \ \forall\, x_1, x_2 \in \mathbb{R}$

5. $\dfrac{1}{n + 1} < \ln\left(1 + \dfrac{1}{n}\right) < \dfrac{1}{n}, \ n > 1$

6. Using Cauchy's theorem applied to the functions:

$f(t) = (1 + t)\ln(1 + t)$, $g(t) = \operatorname{arctg} t$,

$f, g : [0, x] \longrightarrow \mathbb{R}$

show that:

$$\ln(1 + x) > \frac{\operatorname{arctg} x}{1 + x}, \text{ for } x > 0.$$

III. Using the method of the minimum, demonstrate the inequalities:

1. $2 \le 2x \cdot \arcsin x + 2\sqrt{1 + x^2} -$

$- x^2 < \pi - 1 , \quad x \in (-1,1)$

2. $-3 \le 2\cos x - \cos 2x \le \dfrac{3}{2}, \ x \in \mathbb{R}$

3. $\ln\left(1 + 2x + 2x^2\right) > -\dfrac{1}{2}, \ x \in \mathbb{R}$

4. $x - \dfrac{x^3}{6} < \sin x < x , \quad x > 0$

5. $x - \dfrac{x^2}{2} < \ln(1 + x) < x , \quad x > 0$

6. $\sin x \ge \dfrac{x}{x + 1}, \ x \in \left[0 , \dfrac{\pi}{2}\right]$

7. $2\left(1 + x^{n+1}\right)^n \ge \left(1 + x^n\right)^{n+1},$

$x > 0, \ n \in \mathbb{N}$

IV. Without calculating the integrals, show that:

1. $\dfrac{1}{3} \le \displaystyle\int_4^7 \frac{x - 3}{x + 5}\, dx \le 1$

2. $1 \le \displaystyle\int_0^1 e^{x^2} dx \le e$

3. $\dfrac{16}{3} \le \displaystyle\int_4^6 \frac{x^2}{x + 2}\, dx \le 9$

4. $\sqrt{10} \le \displaystyle\int_{-4}^{-3} \sqrt{x^2 + 1}\, dx \le \sqrt{17}$

5. $0 \le \displaystyle\int_{-\frac{1}{2}}^{0} x \ln\left(1 + x^2\right) dx \le \frac{1}{4} \ln \frac{4}{3}$

$$6.\ \int_0^{\frac{\pi}{2}} e^{-\sin x}\,dx < \frac{\pi}{2}\left(1 - \frac{1}{e}\right)$$

V. Without calculating the integral, show which one of the following integrals has the highest value:

$$1.\ \int_1^2 \ln(1 + x)\,dx \quad \text{or} \quad \int_1^2 \frac{x}{x + 1}\,dx$$

$$7.\ \int_0^{\ln \pi} \sin e^x\,dx < \pi - 1$$

$$2.\ \int_2^{10} x \cdot \operatorname{arctg} x\,dx \quad \text{or} \quad \int_2^{10} \ln\left(1 + x^2\right)\,dx$$

$$3.\ \int_0^{\frac{\pi}{2}} \sin^n x\,dx \quad \text{or} \quad \int_0^{\frac{\pi}{2}} \sin^{n+1} x\,dx$$

Indication: Using the method of the minimum, we prove inequalities of the form $f_1(x) < f_2(x)$ or $f_1(x) > f_2(x)$ on the considered intervals. We then integrate the obtained inequality.

VI. 1. If $f: I \to \mathbb{R}$ is convex, then for any $x_1, x_2, \dots, x_n \in I$, we have the inequality (Jensen's inequality):

$$f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right) < \frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n}$$

(Manual)

2. Using Jensen's inequality applied to the convex function $f(x) = e^x$, prove the inequalities of the averages:

$$\sqrt[n]{a_1 \cdot a_2 \cdot \dots \cdot a_n} < \frac{a_1 + a_2 + \dots + a_n}{n}$$

3. Show that for any $x_1, x_2, \dots, x_n \in [o, \frac{\pi}{2}]$, we have:

$$\sin \frac{x_1 + x_2 + \dots + x_n}{n} >$$

$$> \frac{\sin x_1 + \sin x_2 + \dots + \sin x_n}{n}$$

$$\cos \frac{x_1 + x_2 + \dots + x_n}{n} >$$

$$> \frac{\cos x_1 + \cos x_2 + \dots + \cos x_n}{n}$$

What are the maximum domains that contain zero and in which these inequalities take place? Give examples of other domains, that don't contain the origin and that host inequalities like those illustrated above. In what domains do inverse inequalities take place?

4. Using the concavity of the logarithmic function, prove that for any $x_1, x_2, \dots, x_n > 0$ we have:

$$x_1 \cdot x_2 \cdot \dots \cdot x_n < \left[\frac{x_1 + x_2 + \dots + x_n}{n}\right]^n$$

5. Show that the function $f(x) = x^\alpha$, $\alpha > 1$, $x > 0$ is convex. Using this property show that for any non-negative numbers $x_1, x_2, \dots, x_n$, the following inequality takes place:

$$(x_1 + x_2 + \dots + x_n)^\alpha < n^{\alpha-1}(x_1^\alpha + x_2^\alpha + \dots + x_n^\alpha)$$

6. Applying Jensen's inequality to the function $f(x) = x^2$, show that for any $x_1, x_2, \dots, x_n \in \mathbb{R}$, we have:

$$(x_1 + x_2 + \dots + x_n)^2 < n \cdot (x_1^2 + x_2^2 + \dots + x_n^2).$$

What analogous inequality can be deduced from the convex function $f(x) = x^3$, for $x > 0$? And from the concave function $f(x) = x^3$, for $x \leq 0$?

7. Applying Jensen's inequality for the convex function $f(x) = \frac{1}{x}$, for $x > 0$, show that, if:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

is a polynomial with all real and positive roots, we have:

$$n^2 \cdot \frac{a_n}{a_{n-1}} > \frac{a_1}{a_0} .$$

What can be said if the polynomial has all the roots real and negative?

SOLUTIONS:

By comparing the right member of the required inequality with the right member of Jensen's inequality, we deduce that the points $x_1, x_2, \dots, x_n$ are deduced from the conditions:

$$f(x_1) = a_1 , \ f(x_2) = a_2 , \ \dots , \ f(x_n) = a_n$$

We have:

$e^{x_1} = a_1 \Rightarrow x_1 = \ln a_1$ **and, generally,** $x_i = \ln a_i$

With these points, Jensen's inequality, for $f(x) = e^x$, becomes:

$$e^{\frac{\ln a_1 + \ln a_2 + \dots + \ln a_n}{n}} < \frac{a_1 + a_2 + \dots + a_n}{n}$$

Making the calculations in the left member we obtain the inequality of the averages.

3. On the interval $[0, \frac{\pi}{2}]$, the functions $\sin x$ and $\cos x$ are concave. The biggest interval that contains the origin and in which $\sin x$ is concave, is the interval $[0, \pi]$. On $[\pi, 2\pi]$, for example, the same function is convex, so the inequality from the enunciation is changing.

VII. Primitives

Connections with other notions specific to functions

Definition : The function $f: I \to \mathbb{R}$ has primitives on the interval I if there exists a function $F: I \to \mathbb{R}$, differentiable and $F'(x) = f(x)$ on I.

It is known that two primitives differ through a constant:

$$F_1(x) = F_2(x) + C$$

The set of all primitives is called an undefined integral of the function f and is noted by:

$$\int f(x)\, dx$$

In order to enunciate **the method employed to show that a function has primitives and the method to show that a function doesn't have primitives,** we recall some of the connections that exist between the main notions of calculus studied in high school, relative to functions:

1. THE LIMIT
2. THE CONTINUITY
3. THE DERIVABILITY
4. DARBOUX'S PROPERTY
5. THE PRIMTIVE
6. THE INTEGRAL

The connections between these notions are given by the following properties:

P_1) Any continuous function in a point x_0 has a limit in this point:

$$C \longrightarrow L$$

P_2) Any differentiable function in a point x_0 is continuous in this point:

$$D \longrightarrow C$$

P_3) Any continuous function on $[a, b]$ has Darboux's property on $[a, b]$:

$$C \longrightarrow \text{Darb.}$$

P_4) Any continuous function on $[a, b]$ has primitives on $[a, b]$:

$$C \longrightarrow P$$

P_5) Any continuous function on $[a, b]$ is integrable on $[a, b]$:

$$C \longrightarrow I$$

P_6) Any function that has primitives on $[a, b]$ has Darboux's property on $[a, b]$:

$$P \longrightarrow \text{Darb.}$$

P_7) If a function has Darboux's property on $[a, b]$, then in any point x_0 in which the limit (the lateral limit) exists, it is equal to $f(x_0)$.

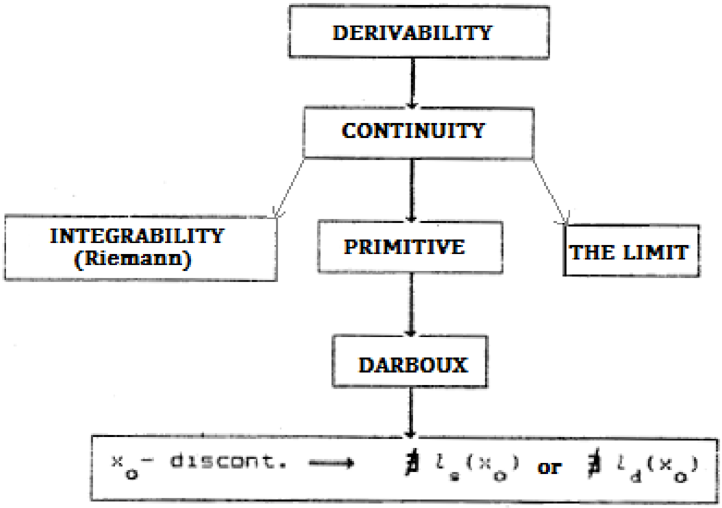
Consequences: 1. A function with Darboux's property (so a function that admits primitives) can't have an infinite limit (lateral limit) in a point x_0.

The functions for which at least one lateral limit is infinite or different from $f(x_0)$ doesn't have primitives.

2. If f has Darboux's property on $[a, b]$ and x_0 is a discontinuity point, AT LEAST ONE LATERAL LIMIT DOESN'T EXIST.

$$\text{Darb.} + x_0 \text{ discont.} \Longrightarrow \nexists\, l_s(x_0) \text{ or } \nexists\, l_d(x_0)$$

Due to the implications stated above, we can form the *Table*:



OBSERVATION: If x_0 is a discontinuity point for f, one of the following situations is possible:

(1) the lateral limits exist and they are finite

(2) a lateral limit is finite, the other is infinite

(3) both lateral limits are infinite

(4) one lateral limit is infinite, the other doesn't exist

(5) one lateral limit is finite, the other doesn't exist

(6) both lateral limits don't exist

FROM ALL THE FUNCTIONS DISCONTINUED IN AT LEAST ONE POINT, THE ONLY ONES THAT CAN HAVE PRIMITIVES ARE THOSE FOR WHICH AT LEAST ONE LATERAL LIMIT DOESN'T EXIST (AND THE OTHER, IF IT IS EXISTS, IS FINITE).

Also, let's observe that, although in case (1) the function doesn't have primitives on $[a, b]$ (doesn't have Darboux's property), still, if, moreover, the lateral limits are equal, then f has primitives on $[a, b]\backslash\{x_0\}$ through the function f_p – the extension through continuity of $f's$ restriction to the domain $[a, b]\backslash\{x_0\}$.

Example:

$$f(x) = \begin{cases} a & x \leq 0 \\ \sin\frac{1}{x} & x > 0 \end{cases}$$

has Darboux's property for any $a \in [-1,1]$, but doesn't have primitives if $a \neq 0$.

Indeed, assuming *ad absurdum* that f has primitives, let F be one of its primitives. Then F must be of the form:

$$F(x) = \begin{cases} a \cdot x + c_1 & x \leq 0 \\ \int \sin\frac{1}{x}\,dx + c_2 & x > 0 \end{cases}$$

In order to calculate $\int \sin\frac{1}{x}\,dx$, we observe that $\sin\frac{1}{x}$ comes from the derivation of the function $x^2\cos\frac{1}{x}$. Indeed,

$$\left(x^2\cos\frac{1}{x} \right) = 2x \cdot \cos\frac{1}{x} + \sin\frac{1}{x}$$

so:

$$\sin\frac{1}{x} = \left(x^2\cos\frac{1}{x} \right) - 2x \cdot \cos\frac{1}{x} \quad \textbf{and:}$$

$$\int \sin\frac{1}{x}\,dx = \int\left(x^2\cos\frac{1}{x} \right)'dx - 2\int x \cdot \cos\frac{1}{x}\,dx =$$

$$= x^2\cos\frac{1}{x} - 2\int x\cos\frac{1}{x}\,dx .$$

The function $g(x) = x\cos\frac{1}{x}$ has primitives on $(0,\infty)$ because it is continuous on this interval, but doesn't have primitives on $[0,\infty)$ (we aim for the interval to be closed at zero in order to study the continuity and derivability od F).

We observe that, because $\lim\limits_{\substack{x \to 0 \\ x > 0}} g(x) = 0$, we can consider its extension through continuity:

$$g_p(x) = \begin{cases} x \cdot \cos \frac{1}{x} & x > 0 \\ 0 & x = 0 \end{cases}$$

This function, being continuous on $[0, \infty)$, has primitives on this interval. Let G be one of these primitives. Then:

$$\int \sin \frac{1}{x} dx = x^2 \cos \frac{1}{x} - 2 \cdot G(x) \quad \textbf{and:}$$

$$F(x) = \begin{cases} \alpha \cdot x + C_1 & x < 0 \\ x^2 \cos \frac{1}{x} - 2 \cdot G(x) + C_2 & x > 0 \end{cases}$$

We state the condition that F be continuous in $x_0 = 0$. We have:

$$F(0) = l_s(0) = C_1 \ , \ l_d(0) = -2 \cdot G(0) + C_2 \ ,$$

so we must have $C_1 = C_2 + 2.G(0)$ and so, for the primitive, we obtain the following formula:

$$F(x) = \begin{cases} a \cdot x + C_1 & x < 0 \\ x^2 \cos \frac{1}{x} - 2 \cdot G(x) + 2 \cdot G(0) + C_1 & x > 0 \end{cases}$$

We state the condition that F be differentiable in zero:

$$F'_s(0) = \lim_{\substack{x \to 0 \\ x < 0}} \frac{F(x) - F(0)}{x} = a \ ,$$

$$F'_d(0) = \lim_{\substack{x \to 0 \\ x > 0}} \frac{F(x) - F(0)}{x} =$$

$$= \lim_{\substack{x \to 0 \\ x > 0}} \frac{x^2 \cos \frac{1}{x} - 2G(x) + 2G(0) + C_1 - C_1}{x}$$

$$= -2 \lim_{\substack{x \to 0 \\ x > 0}} \frac{G(x) - G(0)}{x} = -2G'(0)$$

because G is differentiable in zero. We have $G'(0) = g_P(0) = 0$ (G is a primitive of g_P), so $F_d'(0) = 0$. Consequently, F is not differentiable in zero if $a \neq 0$.

Observation: For the study of derivability we have mentioned two methods: using the definition and using the corollary of Lagrange's theorem. Let's observe that in the example from above, we can't use the corollary of Lagrange's theorem because $\lim_{\substack{x \to 0 \\ x > 0}} F'(x)$ doesn't exist.

The table at page 193 allows us to formulate methods to show that a function has primitives and methods to show that a function doesn't have primitives.

We will enunciate and exemplify these methods, firstly with exercises from the calculus manual, XII grade, in order to highlight the necessity of familiarizing oneself with the different notions frequently used in high school manuals.

(A) Methods to show that a function has primitives

The first three methods are deduced from the table, using implications of the form $a \to b$:

(EP_1) We show that the function is differentiable.

(EP_2) We show that the function is continuous.

(EP_3) We construct the primitive.

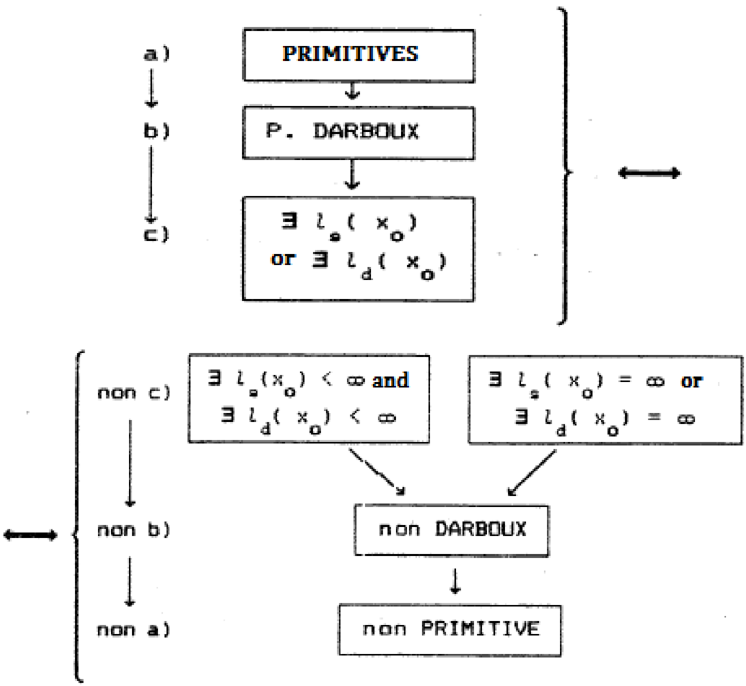
(EP_4) We show that the function is a sum of functions that admit primitives.

Keep in mind that method EP_2 is preferred over EP_1 because derivability is easier to prove that continuity, and the method EP_3 is used only if the function doesn't have a limit in a discontinuity point (this is the only situation where it can still have primitives).

(B) Methods to show that a function doesn't have primitives

The first three methods are deduced from the table, using the equivalence: $a \rightarrow b \leftrightarrow non\ b \longrightarrow non\ a$.

We therefore have:



This way, we deduce the following methods to show that a function doesn't have primitives:

(NP_1) We show that **in a discontinuity point**, both lateral limits exist and are finite or at least one lateral limit is infinite..

(NP_2) We show that the function doesn't have Darboux's property..

(NP_3) We assume *ad absurdum* that f has primitives and we obtain a contradiction.

(NP_4) If f is the sum of two function, one of them admitting primitives, the other not admitting primitives, it follows that f doesn't have primitives.

Examples:

$$(EP_2)\quad f(x) = \begin{cases} \frac{1}{x^5} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

has primitives because f is continuous.

$$(EP_3)\quad f(x) = \begin{cases} 2x \cdot \sin\frac{1}{x} - \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Solution: We observe that $2x.\,sin\frac{1}{x} - cos\frac{1}{x}$ comes from the derivation of the function $x^2 sin\frac{1}{x}$, so, a primitive of f must have the form:

$$F(x) = \begin{cases} x^2 \sin\frac{1}{x} + C & x \neq 0 \\ \alpha & x = 0 \end{cases}$$

We state the condition that F be continuous in zero:

$$l_s(0) = l_d(0) = \lim_{x \to 0} \left(x^2 \sin\frac{1}{x} + C\right) = C,\ F(0) = \alpha$$

so $\alpha = C$ and we have:

$$F(x) = \begin{cases} x^2 \sin\frac{1}{x} + C & x \neq 0 \\ C & x = 0 \end{cases}$$

We state the condition that F be differentiable in zero:

$$F'(0) = \lim_{x \to 0} \frac{F(x) - F(0)}{x} =$$

$$= \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x} + C - C}{x} = \lim_{x \to 0} x \cdot \sin \frac{1}{x} = 0$$

(something bounded multiplied with something that tends to zero, tends to zero), so F is differentiable in zero. But we must also have $F'(0) = f(0)$, equality which is also met, so f has primitives.

$$(EP_4) \quad f(x) = \begin{cases} \cos^2 \frac{1}{x} & x \neq 0 \\ \frac{1}{2} & x = 0 \end{cases}$$

We have:

$$f(x) = \begin{cases} \dfrac{1 - \cos \frac{2}{x}}{2} & x \neq 0 \\ \dfrac{1}{2} & x = 0 \end{cases} = g(x) + h(x),$$

where $g(x) = \frac{1}{2}$ and:

$$h(x) = \begin{cases} (\cos \frac{2}{x})/2 & x \neq 0 \\ 0 & x = 0 \end{cases},$$

and the functions g and h are primitives.

(NP_1) a) $f(x) = [x] - x$ doesn't have primitives on $\mathbb{R}$ because in points $x_0 = n$ is discontinuous and both lateral limits are finite.

$$\text{b)} \quad f(x) = \begin{cases} 0 & x < 0 \\ \sin \frac{1}{x} - \frac{1}{x} \cos x & x > 0 \end{cases}$$

doesn't have primitives because $\lim\limits_{\substack{x \to 0 \\ x > 0}} f(x) = -\infty$.

(NP_2) The functions in which this method is used in the manual are of the form:

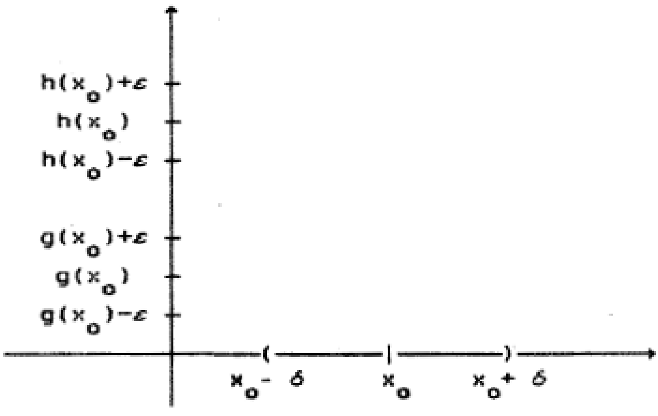
$$f(x) = \begin{cases} g(x) & x \in \mathbb{Q} \\ h(x) & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

with g and h, continuous functions, $g \neq h$. We will prove on this general case that f doesn't have Darboux's property. The demonstration can be adapted to any particular case.

We must prove there exists an interval whose image through f isn't an interval. We use the two conditions from the hypothesis.

Solving the exercise: We must prove there exists an interval whose image through f isn't an interval, using the two conditions from the hypothesis.

From the condition $g \neq h$ we deduce there exists at least one point x_0 in which $g(x_0) \neq h(x_0)$. Then, for ε small enough, the intervals $(g(x_0) - \varepsilon, g(x_0) + \varepsilon)$ and $(h(x_0) - \varepsilon, h(x_0) + \varepsilon)$ are disjunctive (see the *Figure* below).



From the conditions g and h continuous in x_0 we deduce:

$$\forall \varepsilon > 0 \quad \exists \delta_1 > 0 \quad \forall x \in \mathbb{R} \quad |x - x_0| < \delta_1 \Rightarrow$$

$$\Rightarrow |g(x) - g(x_0)| < \varepsilon$$

$$\forall \varepsilon > 0 \quad \exists \delta_2 > 0 \quad \forall x \in \mathbb{R} \quad |x - x_0| < \delta_2 \Rightarrow$$

$$\Rightarrow |h(x) - h(x_0)| < \varepsilon$$

Considering, as we said ε small enough for the intervals $(g(x_0) - \varepsilon, g(x_0) + \varepsilon)$ and $(h(x_0) - \varepsilon, h(x_0) + \varepsilon)$ to be disjunctive, and $\delta = \min\{\delta_1, \delta_2\}$, it follows that the image through the function f of the interval $(x_0 - \delta, x_0 + \delta)$ isn't an interval, because:

$$x \in (x_0 - \delta, x_0 + \delta) \cap \mathbb{Q} \Rightarrow$$

$$\Rightarrow f(x) = g(x) \in (g(x_0) - \varepsilon, g(x_0) + \varepsilon)$$

$$x \in (x_0 - \delta, x_0 + \delta) \cap (\mathbb{R} \setminus \mathbb{Q}) \Rightarrow$$

$$\Rightarrow f(x) = h(x) \in (h(x_0) - \varepsilon, h(x_0) + \varepsilon).$$

(NP_3) This method presupposes the same steps as in method EP_3.

(a) we look for primitives for $f's$ branches and we obtain a first expression of F;

(b) we state the condition that F be continuous in the connection point (points) between the branches;

(c) we state the condition that F be differentiable in these points:

- if F is differentiable and $F'(x_0) = f(x_0)$, we have covered method EP_3.
- if F isn't differentiable in x_0 or $F'(x_0) \neq f(x_0)$, we have covered method (NP_3).

$$(NP_4)\ f(x) = \begin{cases} x \cdot \sin \frac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

doesn't have primitives on ℝ.

Indeed, we have $f(x) = g(x) + h(x)$ with:

$$g(x) = \begin{cases} x \cdot \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$ **and**

$$h(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

and g is continuous so it has primitives and h has lateral limits, finite in zero, so it doesn't have primitives. If, let's say by absurd, f had primitives, it would follow that $h(x) = f(x) - g(x)$ has primitives.

Exercises

I. Using the method (EP_2), show that the following functions have primitives in ℝ:

1. $f(x) = \max(x, x^2)$
2. $f(x) = \begin{cases} a \cdot \ln(3 - x) & x < 1 \\ \dfrac{2^x - 2}{x - 1} & x > 1 \end{cases}$
3. $f(x) = \left| \left[x - \frac{1}{2} \right] - x \right|$
4. $f(x) = \min\limits_{x \in \mathbb{Z}} | x - k |$

5. $f(x) = \begin{cases} \ln(1 - x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

6. $f(x) = \begin{cases} |x|^x & x \neq 0 \\ a & x = 0 \end{cases}$

7. $f(x) = \begin{cases} x \cdot \left| \ln|x| + x \right| & x \neq 0 \\ a & x = 0 \end{cases}$

8. $f(x) = \begin{cases} \dfrac{\sin(n \cdot \arccos x)}{\sqrt{1 - x^2}} & x \in (-1,1) \\ n & x = 1 \text{ or } x = -1 \end{cases}$

II. Using methods $(EP_3) - (NP_3)$, study if the following functions have primitives:

1. $f(x) = \begin{cases} 3x - 1 & x \leq 2 \\ 2x + 1 & x > 2 \end{cases}$

2. $f(x) = \begin{cases} \dfrac{1}{\cos^2 \frac{1}{x}} & x \neq 0 \\ a & x = 0 \end{cases}$

3. $f(x) = \max(1 - x, \ln x)$

4. $f(x) = \begin{cases} \cos \frac{1}{x} & x < 0 \\ \operatorname{arctg}(a + x) & x \geq 0 \end{cases}$

5. $f(x) = \begin{cases} \operatorname{arctg} \frac{1}{x} & x < 0 \\ \operatorname{arctg} x & x \geq 0 \end{cases}$

6. $f(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{x} - \frac{1}{x} \cos x & x > 0 \end{cases}$

$$7.\ f(x) = \begin{cases} g'(x)\cdot\sin\dfrac{1}{g(x)} & x \neq 0 \\ a & x = 0 \end{cases}$$

if $g: \mathbb{R} \to \mathbb{R}$ and if it satisfies: $g(x) = 0 <=> x = 0$.

$$8.\ f(x) = \begin{cases} |x - a|\cdot\sin\dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$9.\ f(x) = \begin{cases} \sin(\sin\dfrac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

10. The product from a function h that admits primitives and a function g differentiable with the continuous derivative is a function that admits primitives.

Indications: If H is the primitive of h, then $H.g'$ is continuous, so it admits primitives. Let G be one of its primitives. We have $(H.g - G)' = h.g$.

$$11.\ f(x) = \begin{cases} g(x)\cdot\sin\dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

g being a differentiable function with a continuous derivative on $\mathbb{R}$.

III. Using method (NP_1), show that the following functions don't have primitives:

$$1.\ f(x) = \begin{cases} \operatorname{arctg}\dfrac{1}{x} & x < 0 \\ \operatorname{arctg} x & x > 0 \end{cases}$$

$$2.\ f(x) = \begin{cases} 0 & x \leq 0 \\ \sin\dfrac{1}{x} - \dfrac{1}{x}\cos x & x > 0 \end{cases}$$

$$3.\ f(x) = \begin{cases} \inf\limits_{t \leq x}(t^2 - t + 1) & x < \frac{1}{2} \\ \sup\limits_{t \geq x}(-t^2 + t - 3) & x > \frac{1}{2} \end{cases}$$

$$4.\ f(x) = \begin{cases} x - \frac{1}{x}\sin x & x \neq 0 \\ 0 & x = 0 \end{cases}$$

IV. Using method (NP_2), show that the following functions don't have primitives:

$$1.\ f(x) = \begin{cases} 3x & x \in \mathbb{Q} \\ 2x^2 + 1 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

$$2.\ f(x) = \begin{cases} \sin x & x \in \mathbb{Q} \\ \cos x & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

$$3.\ f(x) = \begin{cases} \ln x & x \in \mathbb{Q} \\ e^x & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

$$4.\ f(x) = \begin{cases} \sqrt{x} & x \in \mathbb{Q} \\ \sqrt[3]{x} & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

$$5.\ f(x) = \begin{cases} x \cdot [x] & x \in \mathbb{Q} \\ -x^2 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Answer:

1. The functions $g(x) = 3x$ and $h(x) = 2x^2 + 1$ are continuous on ℝ and $g \neq h$. We will show that f doesn't have Darboux's property. Let x_0 be a point in which we have $(x_0) = h(x_0)$, for example $x_0 = 2$. We have $g(2) = 6, h(2) = 9$.

$h - continuous\ in\ x_0 = 2 <=>$

$$<=> \left[\forall \varepsilon > 0 \quad \exists \delta_1 > 0 \quad \forall x \in \mathbb{R}, \right.$$

$$\left. |x - 2| < \delta_1 => |3x - 6| < \varepsilon \right]$$

$g - continuous\ in\ x_0 = 2 <=>$

$$<=> \left[\forall \varepsilon > 0 \quad \exists \delta_2 > 0 \quad \forall x \in \mathbb{R}, \right.$$

$$\left. |x - 2| < \delta_2 => |2x^2 + 1 - 9| < \varepsilon \right]$$

Let there be ε so that the intervals:

$(6 - \varepsilon, 6 + \varepsilon)$ **and** $(9 - \varepsilon, 9 + \varepsilon)$

are disjunctive. For example $\varepsilon = 1$ and let there be $\delta = \min(\delta_1, \delta_2)$. Then the image of the interval $(2 - \delta, 2 + \delta)$ through f isn't an interval because for:

$x \in \mathbb{Q} \cap (2 - \delta, 2 + \delta)$

we have $f(x) = 3x \in (5,7)$ and for:

$x \in (\mathbb{R} \setminus \mathbb{Q}) \cap (2 - \delta, 2 + \delta)$

we have $f(x) = 2x^2 + 1 \in (8,10)$.

V. Using methods $(EP_4) - (NP_4)$, study if the following functions admit primitives:

$$1. \quad f(x) = \begin{cases} \cos^3 \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Indication:

$$\cos \frac{3}{x} = 4 \cdot \cos^3 \frac{1}{x} - 3 \cos \frac{1}{x},$$

so $f(x) = g(x) + h(x)$, with:

$$g(x) = \begin{cases} \frac{1}{4} \cos \frac{3}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \quad \textbf{and}$$

$$h(x) = \begin{cases} 3 \cdot \cos \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$2.\ f(x) = \begin{cases} \frac{1}{1 + \sin x} \sin \frac{1}{x} \\ a \end{cases}$$

$x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \setminus \{0\}$

$x = 0$

Indication:

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \setminus \{0\} \\ 0 \end{cases} +$$

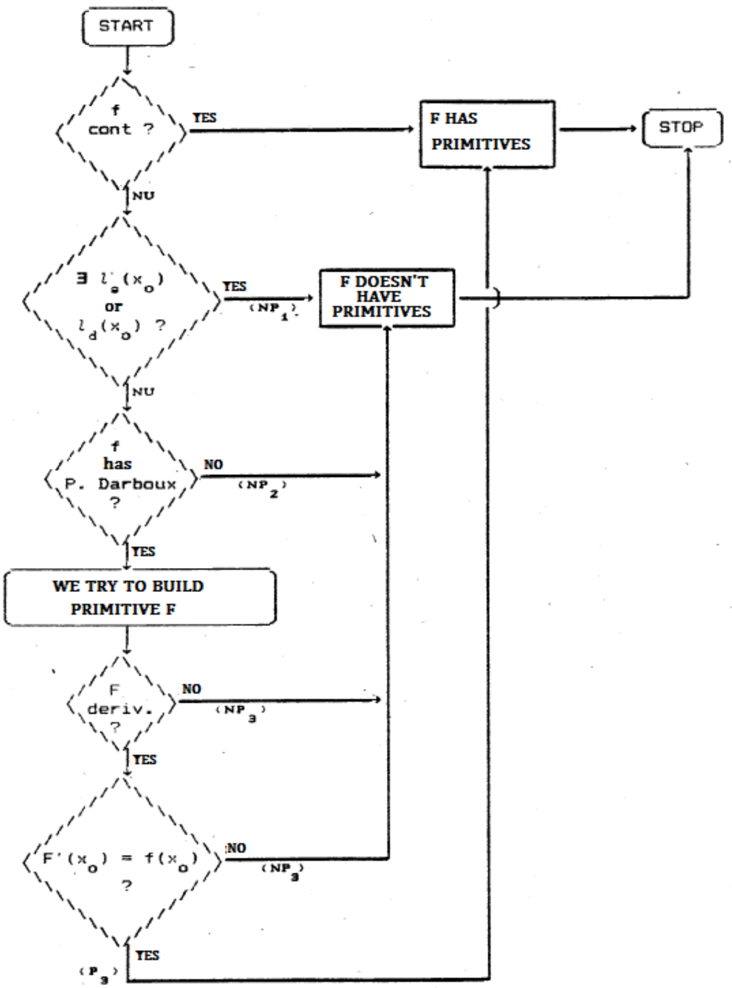
$$+ \quad \begin{cases} \frac{1}{1 + \sin x} \sin \frac{1}{x} \\ 0 & x = 0 \end{cases}$$

We obtain $a = 0$.

$$3.\ f(x) = \begin{cases} \sin^n \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}, \quad n \in \mathbb{N}.$$

Logical Scheme for Solving Problems

Solving problems that require studying if a function has primitives can be done according to the following logical scheme:



VIII. Integration

An integral is defined with the help of a limit. It is much easier to understand the theoretical aspects regarding the integral if one has understood why it is necessary to renounce the two broad categories of limits considered up to this point:

1. the limit when x tends to x_0,
2. the limit when n tends to infinity,

and a new category of limits is introduced:

3. the limit when the division's norm tends to zero.

The specifications that follow are aimed exactly at clarifying this aspect.

The theory of the integral has appeared out of a practical necessity, in order to calculate the area that lays between the graphic of a function and the axis $0x$.

Given a bounded function and (for now) non-negative $f: [a, b] \to \mathbb{R}$ we can approximate the desired area with the help of certain rectangles having the base on $0x$.

For this, we consider the points:

$$a = x_0 < x_1 < \ldots < x_n = b \quad .$$

The set of these points form a division of the interval $[a, b]$ and they are noted with Δ:

$$\Delta = \{ x_0 , x_1 , x_2 , \ldots , x_n \}$$

The vertical strips made using the points x_i have an area as hard to calculate as the initial area, because of the superior outline. We obtain rectangles if we replace the superior outlines with horizontal segments.

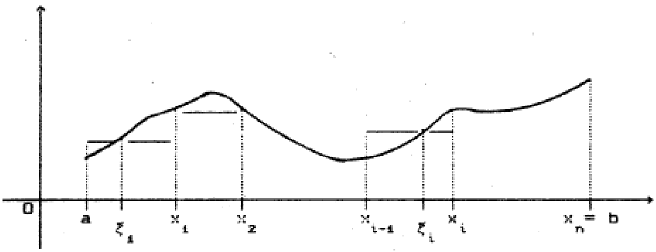


Fig. 8.1

In order that the obtained approximation be reasonable, it is only natural that these horizontal segments meet the graphic of the function. We observe that such a horizontal is uniquely determined by a point ξ_i situated between x_{i-1} and x_i. In this way, we have obtained the rectangles with which we aimed to approximate the desired area.

The areas of the rectangles are:

$$(x_1 - x_0) \cdot f(\xi_1)\ ,\ (x_2 - x_1) \cdot f(\xi_2)\ ,$$

$$,\ \ldots\ ,\ (x_n - x_{n-1}) \cdot f(\xi_n)$$

The sum of these areas is called the **Riemann sum** attached to the function f, to the division Δ and to the intermediary points ξ_i. This sum:

$$\sigma_\Delta(f,\xi) = (x_1 - x_0)f(\xi_1) + (x_2 - x_1)f(\xi_2) + \ldots$$

$$\ldots + (x_n - x_{n-1})f(\xi_n) = \sum_{i=1}^{n}(x_i - x_{i-1})f(\xi_i)$$

approximates the desired area.

The necessity of introducing the third category of limits is due to **the necessity that the approximation be as accurate as possible**.

And now, a question: Is the approximation better when:

(a) the rectangles increase in number, or when

(b) the rectangles are narrower and narrower?

We observe that the rectangles “increase in number” if and only if n tends to infinity, but making n tend to infinity we do not obtain a “better approximation”. Indeed, if we leave the first rectangle unchanged, for example, and raising the number of points from its location to its right as much as we want (making n tend to infinity), the approximation remains coarse. *The approximation becomes much “better” if the rectangles are “thinner and thinner”.*

In order to concretize this intuitive intuition, we observe that the rectangles are thinner and thinner if the “thickest of them” becomes thinner and thinner.

The biggest thickness of the rectangles determined by a division Δ is called the norm of the division Δ.

$$\|\Delta\| = \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\} =$$

$$= \max_{i=1,n}(x_i - x_{i-1})$$

Therefore, “better approximations” <=> “thinner and thinner rectangles” <=> “the biggest thickness tends to zero” <=> “the norm of Δ tends to zero” <=> $||\Delta|| \to 0$.

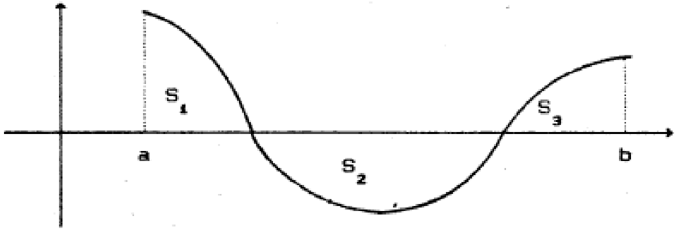
The area of the stretch situated between the graphic of the function f and the axis $0x$ is, therefore:

$$\lim_{\|\Delta\| \to 0} \sigma_\Delta(f, \xi)$$

This limit is noted with $\int_a^b f(x)\,dx$ and it is called the integral of function f on the interval $[a, b]$.

$$\int_a^b f(x)\,dx = \lim_{\|\Delta\| \to 0} \sigma_\Delta(f, \xi)$$

For functions that are not necessarily non-negative on $[a, b]$, the integral is represented by the difference between the area situated above the axis $0x$ and the area situated below the axis $0x$.



$$\int_a^b f(x)\,dx = S_1 - S_2 + S_3$$

Consequently, in order to obtain the area of the stretch situated between the graphic and the axis $0x$, in such a case, we have to take into consideration the module of f:

$$A = \int_a^b |f(x)|\,dx$$

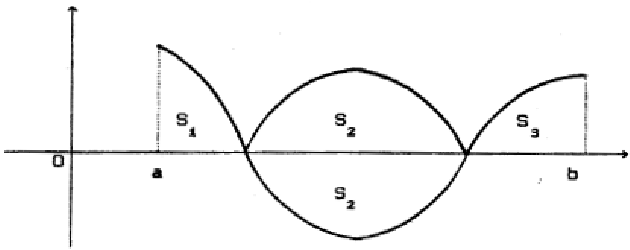


Fig. 8.3

Coming back to the affirmations (a) and (b) from above, we observe that: “the rectangles are thinner and thinner” => “the rectangles increase in number”, namely $(b) => (a)$.

In other words, $\left||\Delta|\right| \to 0 = n \to \infty$.

Reciprocally, this statement isn’t always true. That is why **we cannot renounce $\left||\Delta|\right| \to 0$ in favor of $n \to \infty$.** Nevertheless, there is a case where the reciprocal implication is true: when the points of the division are equidistant.

$n \to \infty => \left||\Delta|\right| \to 0$ for equidistant points.

In this case, the rectangles have the same dimension of the base, namely $x_i - x_{i-1} = \frac{b-a}{n}$ for any $i = 1,2, \dots, n$, and the Riemann sum becomes:

$$\sigma_\Delta(f,\xi) = \frac{b-a}{n}\left(f(\xi_1) + f(\xi_2) + \dots + f(\xi_n) \right) =$$

$$= \frac{b-a}{n} \sum_{i=1}^{n} f(\xi_i)$$

For divisions such as these, the following equivalence takes place:

$$||\Delta|| \to 0 <=> n \to \infty$$

that allows, as it was revealed by *Method 16* for the calculation of limits, for the utilization of the definition of the integral for the calculation of sequences' limits.

Before we enumerate and exemplify the methods for the study of integration, let's observe that, if **the function is continuous,** among the Riemann sums $\sigma_\Delta(f,\xi)$ that can be obtained by modifying just the heights of the rectangles (modifying just the intermediary points ξ_i), there exists a biggest and smallest Riemann sum, namely the Riemann sum having the highest value is obtained by choosing the points $\xi_i \in [x_{i-1}, x_i]$ for which:

$$f(\xi_i) = \sup \{ f(x) \mid x \in [x_{i-1}, x_i] \} = M_i ,$$

and the Riemann sum having the smallest value is obtained by choosing the points ξ_i for which:

$$f(\xi_i) = \inf \{ f(x) \mid x \in [x_{i-1}, x_i] \} = m_i .$$

By noting with $S_\Delta(f)$ and $s_\Delta(f)$, respectively, these sums (called the superior Darboux sum, the inferior Darboux sum, respectively), we have:

$$S_\Delta(f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \quad \textbf{and}$$

$$s_\Delta(f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \quad \textbf{and}$$

$s_\Delta(f) \leq \sigma_\Delta(f,\xi) \leq S_\Delta(f)$ **no matter what division** Δ **is** (8.1)

The points ξ_i that realizes the supreme, respectively, the infimum of f on the intervals $[x_{i-1}, x_i]$ exists because it is known that a continuous function, on a closed and bounded interval is bounded and it touches its bounds. It is known (see *Theorem 37* from the *Calculus manual, grade XI*) that a function is integrable if and only if:

$$\lim_{\|\Delta\| \to 0} \left(S_\Delta(f) - s_\Delta(f) \right) = 0 \qquad (8.2)$$

Connections between integration and other notions specific to functions

In order to formulate such connections, we fill the list of the propositions $P_1 - P_7$ enunciated in the previous chapter, with the following:

P_8) any function integrable on $[a, b]$ is bounded,

P_9) any function monotonous on $[a, b]$ is integrable,

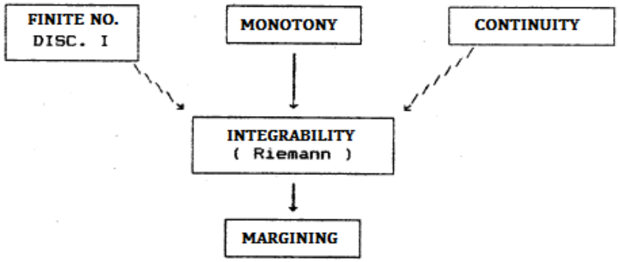
P_{10}) any function continuous on $[a, b]$ is integrable,

P_{11}) any function that has on $[a, b]$ a finite number of discontinuity points of first order is integrable.

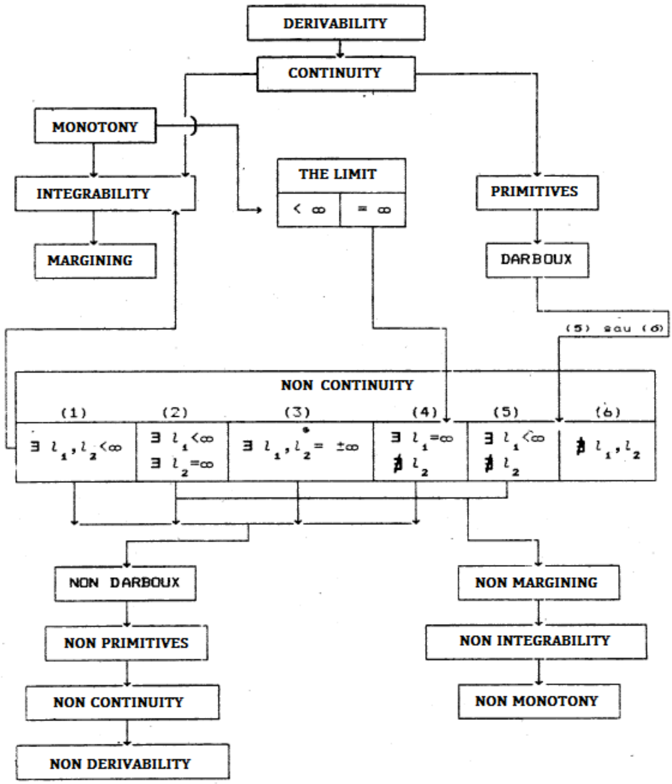
The demonstration of P_{10} follows from the fact that if we modify the values of a function integrable on $[a, b]$, in a finite number of points, we obtain also an integrable function and from the property of additivity in relation to the interval of the integral:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Using the proposition stated above, we can make the following table:



and the table containing implications between all the notions that we have dealt with is the following (l_1 and l_2 being one of the lateral limits):



Methods to show that a function is integrable (Riemann) on $[a, b]$

(I_1) using the definition;

(I_2) we show that it is continuous;

(I_3) we show that it has a finite number of discontinuity points of first order;

(I_4) we show it is monotonic;

(I_5) we show that the function is a sum of integrable functions.

Methods to show that a function is not integrable (Riemann) on $[a, b]$

(N_1) using the definition (we show that $\sigma_\Delta(f, \xi)$ doesn't have a finite limit for $||\Delta|| \to 0$);

(N_2) we show that the function is not bounded;

(N_3) we show that the function is the sum of an integrable function and a non-integrable function.

Examples

(I_1) (a) Using the definition of the integral, show that any differentiable function with the derivative bounded on $[a, b]$ is integrable on $[a, b]$.

Solution: Let $f: [a, b] \to \mathbb{R}$ be differentiable. We have to show that $\sigma_\Delta(f, \xi)$ has a finite limit when $||\Delta|| \to 0$. The given function being differentiable, it is continuous, so it has primitives. Let F be one of its primitives. We can apply Lagrange's theorem to function F, on any interval $[x_{i-1}, x_i]$. There exists therefore $c_i \in [x_{i-1}, x_i]$ so that:

$$F(x_i) - F(x_{i-1}) = f(c_i)(x_i - x_{i-1}) .$$

It follows that:

$$\sigma_\Delta(f,\xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) =$$

$$= \sum_{i=1}^{n} \big(f(c_i) - f(c_i) + f(\xi_i) \big)(x_i - x_{i-1}) =$$

$$= \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) +$$

$$+ \sum_{i=1}^{n} \big(f(\xi_i) - f(c_i)\big)(x_i - x_{i-1}) =$$

$$= \sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1}) \big) +$$

$$+ \sum_{i=1}^{n} \big(f(\xi_i) - f(c_i) \big)(x_i - x_{i-1}) .$$

We calculate the two sums separately:

$$\sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1}) \big) = \big(F(x_1) - F(x_0) \big) +$$

$$+ \big(F(x_2) - F(x_1) \big) + \ldots +$$

$$+ \big(F(x_n) - F(x_{n-1}) \big) =$$

$$= F(x_n) - F(x_0) = F(b) - F(a).$$

We show that the second sum tends to zero when $||\Delta|| \to 0$. For this, we take into consideration that f' is bounded, in other words, $M > 0$ exists, so that $|f'(x) < M|$ for any x from $[a, b]$ and we apply the theorem of Lagrange to the function f, but on the intervals of extremities ξ_i and c_i. There exists θ_i between ξ_i and c_i so that:

$$f(\xi_i) - f(c_i) = (\xi_i - c_i)f'(\theta_i)$$

It follows therefore that:

$$| f(\xi_i) - f(c_i) | = | \xi_i - c_i | \cdot | f'(\theta_i) | \leq$$

$$\leq | \xi_i - c_i | \cdot M \leq \| \Delta \| \cdot M$$

For the second sum, we therefore have:

$$\left| \sum_{i=1}^{n} (f(\xi_i) - f(c_i))(x_i - x_{i-1}) \right| \leq$$

$$\sum_{i=1}^{n} \left| f(\xi_i) - f(c_i) \right| \cdot | x_i - x_{i-1} | \leq$$

$$\leq \sum_{i=1}^{n} | \xi_i - c_i | \cdot M \cdot | x_i - x_{i-1} | \leq$$

$$\leq \sum_{i=1}^{n} \| \Delta \| \cdot M \cdot | x_i - x_{i-1} | =$$

$$= \| \Delta \| \cdot M \cdot \sum_{i=1}^{n} (x_i - x_{i-1}) =$$

$$= \| \Delta \| \cdot M \cdot (b - a) \xrightarrow{\|\Delta\| \longrightarrow 0} 0 .$$

The condition from this exercise's hypothesis is very strong. As it is known, it can be shown that a function that fulfills only the continuity condition is also integrable. Still, the method used in this exercise can be easily adapted to most exercises that require showing that a function is integrable.

(I_1) (b) Show that $f) x_{=\sin} x$ is integrable on any interval $[a, b]$. (*Manual*).

Solution: We will adapt the previous demonstration. The given function is differentiable and has the derivative $f'(x) = \cos x$ bounded. Being differentiable, it is continuous and it has primitives.

Let F be one of its primitives, $F(x) = -\cos x$ and let:

$$\Delta = (a = x_0 , x_1 , \ldots , x_n = b)$$

be a division of $[a,b]$. We apply Langrange's theorem to the function F on each interval $[x_{i-1}, x_i]$. There exists therefore $c_i \in [x_{i-1}, x_i]$ so that:

$$F(x_i) - F(x_{i-1}) = f'(c_i)\cdot(x_i - x_{i-1}) .$$

The function F has a bounded derivative: $|\cos x| < 1$ for any x from $\mathbb{R}$, so for any x from $[a,b]$. We have to show that $\sigma_\Delta(f,\xi)$ has a finite limit when $||\Delta|| \to 0$. We will show that this limit is (as it is known) $F(b) - F(a) = -\cos b + \cos a$.

We have:

$$\sigma_\Delta(f,\xi) = \sum_{i=1}^{n} f(\xi_i)\cdot(x_i - x_{i-1}) =$$

$$= \sum_{i=1}^{n} \left(f(c_i) - f(c_i) + f(\xi_i) \right)(x_i - x_{i-1}) =$$

$$= \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) +$$

$$+ \sum_{i=1}^{n} \left(f(\xi_i) - f(c_i) \right)(x_i - x_{i-1}) =$$

$$= \sum_{i=1}^{n} \left(F(x_i) - F(x_{i-1}) \right) +$$

$$+ \sum_{i=1}^{n} \left(f(\xi_i) - f(c_i) \right)(x_i - x_{i-1}) =$$

$$= F(b) - F(a) + \sum_{i=1}^{n} \left(f(\xi_i) - f(c_i)\right)(x_i - x_{i-1}) .$$

Applying the theorem of Lagrange to the function $f(x) = \sin x$, on the interval determined by the points ξ_i and c_i, we deduce there exists θ_i between ξ_i and c_i so that:

$$\sin \xi_i - \sin c_i = (\xi_i - c_i)\cdot\cos\theta_i \text{, so}$$

$$\left| \sum_{i=1}^{n} \left(f(\xi_i) - f(c_i) \right)(x_i - x_{i-1}) \right| \leq$$

$$< \sum_{i=1}^{n} | f(\xi_i) - f(c_i) | \cdot | x_i - x_{i-1} | =$$

$$= \sum_{i=1}^{n} | \sin \xi_i - \sin c_i | \cdot | x_i - x_{i-1} | =$$

$$= \sum_{i=1}^{n} | \cos \theta_i | \cdot | \xi_i - c_i | \cdot | x_i - x_{i-1} | <$$

$$< \sum_{i=1}^{n} 1 \cdot \| \Delta \| \cdot | x_i - x_{i-1} | =$$

$$= \| \Delta \| \sum_{i=1}^{n} (x_i - x_{i-1}) = \| \Delta \| \cdot (b - a) \xrightarrow[\|\Delta\| \to 0]{} 0 .$$

So: $\lim\limits_{\|\Delta\| \to 0} \sigma_\Delta(f,\xi) = F(b) - F(a) = -\cos b + \cos a .$

(I_2) The function $f: [0,3] \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} \operatorname{arctg} \dfrac{\pi}{2 - x} & \text{if } x \in [0,2) \\ \pi/2 & \text{if } x = 2 \\ (e^{x-2} + x - 2)^{\ln\left(\frac{\pi}{2}\right)^{\frac{1}{2(x-2)}}} & \text{if } x \in (2,3] \end{cases}$$

is integrable. Indeed, it is continuous in any point $x \neq 2$ from the definition domain, being expressed through continuous functions. We study the continuity in $x = 2$:

$$l_s(2) = \lim_{\substack{x \to 2 \\ x < 2}} f(x) = \operatorname{arctg} \frac{\pi}{0+} = \operatorname{arctg}(+\infty) = \frac{\pi}{2}$$

$$l_d(2) = \lim_{\substack{x \to 2 \\ x > 2}} f(x) = \frac{\pi}{2} = f(2) .$$

The function is continuous on the interval $[0,3]$, so it is integrable.

(I_3) $f(x) = x - [x]$, $f: [0, 2\sqrt{3}] \to \mathbb{R}$ can be discontinuous just in the points where $[x]$ is discontinuous. These are of the form

$x = n, n \in \mathbb{Z}$. From these, only points $0,1,2,3,$ are in the function's domain. We study the limit in these points. We have:

$$l_s(1) = l_s(2) = l_s(3) = l_s(3) = 1 \text{ and}$$

$$l_d(0) = l_d(1) = l_d(2) = l_d(3) = 0 ,$$

so the function has a finite number of discontinuity points of first order and is therefore integrable.

(I_4) For the function $f(x) = [x]$ we can say it is integrable either by using the previous criterium, either using the fact that it is monotonic on any interval $[a, b]$.

(I_5) The function $f: [-1,1] \to \mathbb{R}$ through $f(x) = sgn\, x + |x| + [x]$ is integrable because it is the sum of three integrable functions: $f_1(x) = sgn\, x$ is integrable because it has a finite number of discontinuity points (a single point, $x = 0$) and the discontinuity is of the first order; the function $f_2(x) = |x|$ is integrable because it is continuous, and the function $f_3(x) = [x]$ is integrable because it is monotonic (or because it has a single f discontinuity point, of the first order).

$$(NI_1) \quad f(x) = \begin{cases} 0 & x \in [0,1] \cap \mathbb{Q} \\ 1 & x \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q}) \end{cases}$$

isn't integrable because, by choosing in the Riemann sum:

$$\sigma_\Delta(f,\xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

the intermediary points ξ_i, rational, we have: $f(\xi_i) = 0$, so in this case:

$$\sigma_\Delta(f,\xi) = \sum_{i=1}^{n} 0 \cdot (x_i - x_{i-1}) = 0 ,$$

and if ξ_i are irrational, we have:

$$\sum_{i=1}^{n} 1\cdot(x_i - x_{i-1}) = \sum_{i=1}^{n}(x_i - x_{i-1}) = b - a .$$

$$(NI_2) \quad f(x) = \begin{cases} \frac{1}{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

isn't integrable (Riemann) on $[0,1]$ because it is not bounded.

$$(NI_3) \quad f(x) = \begin{cases} x + \frac{1}{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

is the sum of the functions:

$$f_1(x) = x \quad \text{and} \quad f_2(x) = \begin{cases} \frac{1}{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

If f would be integrable, from $f = f_1 + f_2$ it would follow that $f_2 = f - f_1$ so f_2 would be integrable on $[0,1]$, as the sum of two integrable functions.

Exercises

I. Using the definition, show that the following functions are integrable:

1. $f(x) = e^x$
2. $f(x) = \text{tg } x$
3. $f(x) = \sqrt{x^2 + 1}$
4. $f(x) = \arcsin \frac{x}{2}$
5. $f(x) = \cos^2 x$
6. $f(x) = \sin 2x$

II. Study the integration of the functions:

1. $f(x) = \begin{cases} x & x \in [0,1] \\ x^2 & x \in (1,2] \end{cases}$
2. $f(x) = \sup\limits_{t \in [0,1]} \left(t^2 - \frac{1}{2} \right)$

3. $f(x) = \max_{t \in [0,\pi]} (\sin t, \cos t)$

4. $f(x) = \lim_{n \to \infty} \dfrac{x^2 + x \cdot e^{nx}}{1 + e^{nx}}$

5. $f(x) = \begin{cases} x \cdot \left[\dfrac{1}{x}\right] & x \in (0,1] \\ a & x = 0 \end{cases}$

6. $f(x) = \begin{cases} \left[\ln \dfrac{1}{x}\right] & x \in [0,1] \\ 0 & x = 0 \end{cases}$

7. $f(x) = \dfrac{4}{3^n} - x$,

$f:[a,1] \longrightarrow \mathbb{R}$, $a \in (0,1)$

8. The characteristic function of the set $[0,1] \subset \mathbb{R}$.

9. $f(x) = \begin{cases} ax + b & x \in [-1,0] \\ cx + d & x \in (0,1] \end{cases}$

III. For which values of the parameters a and b are the functions below integrable? For which values of the parameters do they have primitives?

1. $f(x) = \begin{cases} \operatorname{arctg}(\ln x) & x \in (0,1] \\ a & x = 0 \end{cases}$

2. $f(x) = \begin{cases} \dfrac{1}{x} & \left[\dfrac{1}{x}\right] = 2n + 1 \\ ax + b & \left[\dfrac{1}{x}\right] = 2n \end{cases}$

3. $f(x) = \begin{cases} \sin \dfrac{1}{x} & x \in (0,1] \\ a & x = 0 \end{cases}$

$$4.\ f(x) = \begin{cases} e^x \cos \frac{1}{x} & x \in (0,1] \\ \alpha & x = 0 \end{cases}$$

$$5.\ f(x) = \begin{cases} 1 & x = \frac{1}{n} \\ 0 & \text{in rest} \end{cases}$$

$$f:[a,b] \longrightarrow \mathbb{R}, \quad a \in (0,1)$$

INDICATIONS:

1. $l_d(0) = -\frac{\pi}{2}$, so for any value of the parameter, the function has only one discontinuity, of first order on $[0,1]$.

2. For any value of $a \in (0,1)$ the function has a limited number of discontinuity points of first order.

4. The function $e^x cos \frac{1}{x}$ comes from the derivation of $x^2 e^x \sin \frac{1}{x}$.

IV. 1. Show that any continuous function on an interval $[a, b]$ is integrable.

2. Adapt the demonstration from the previous point to show that the following functions are integrable on the interval $[0,1]$:

(a) $f(x) = x^2$ (b) $f(x) = \sin x$ (c) $f(x) = e^x$

ANSWERS:

1. We will show that:

$$\lim_{\|\Delta\| \to 0} \left(S_\Delta(f) - s_\Delta(f) \right) = 0$$

We have:

$$S_\Delta(f) - s_\Delta(f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) -$$

$$- \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}).$$

where:

$$M_i = \sup \{ f(x) \mid x \in [x_i, x_{i-1}] \},$$

$$m_i = \inf \{ f(x) \mid x \in [x_i, x_{i-1}] \}.$$

We now use two properties of continuous functions on a closed and bounded interval:

(1) a continuous function on a closed, bounded interval is bounded and it touches its bounds; so, there exists $u_i, v_i \in [a,b]$ so that $M_i = f(u_i), m_i = f(v_i)$.

Therefore:

$$S_\Delta(f) - s_\Delta(f) = \sum_{i=1}^{n} (f(u_i) - f(v_i))(x_i - x_{i-1})$$

In order to evaluate the difference $f(u_i) - f(v_i)$, we use the property:

(2) a continuous function on a closed and bounded interval is uniformly continuous (see *Chapter IV*), namely:

$$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 \quad \forall x_1, x_2 \in [a,b],$$

$$|x_1 - x_2| < \delta_\varepsilon \Longrightarrow |f(x_1) - f(x_2)| < \varepsilon.$$

In order to replace x_1 with u_i and x_2 with v_i, we have to consider a division Δ having a smaller norm than δ_ε (which is possible because we are making the limit for $||\Delta|| \to 0$). Therefore, assuming $||\Delta|| < \delta_\varepsilon$ we have:

$$| f(u_i) - f(v_i) | < \varepsilon,$$

i.e. $M_i - m_i < \varepsilon$ and so:

$$S_\Delta(f) - s_\Delta(f) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) <$$

$$< \sum_{i=1}^{n} \varepsilon(x_i - x_{i-1}) = \varepsilon \sum_{i=1}^{n} (x_i - x_{i-1}) = \varepsilon(b - a).$$

Because ε is arbitrary (small), we deduce that:

$$S_\Delta(f) - s_\Delta(f) \xrightarrow[||\Delta|| \to 0]{} 0$$

so the function is integrable.

2. We show that $S_\Delta(f) - s_\Delta(f)$ tends to zero when $||\Delta||$ tends to zero. We have:

$$S_\Delta(f) - s_\Delta(f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) -$$

$$- \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1})$$

where:

$$M_i = \sup\{x^2 \mid x \in [x_{i-1}, x_i]\} = x_i^2$$

$$m_i = \inf\{x^2 \mid x \in [x_{i-1}, x_i]\} = x_i^2 ,$$

because on the interval $[0,1]$ the function $f(x) = x^2$is increasing. So:

$$S_\Delta(f) - s_\Delta(f) = \sum_{i=1}^{n} (x_i^2 - x_{i-1}^2)(x_i - x_{i-1})$$

The function $f(x) = x^2$ is continuous on $[0,1]$ so it is uniformly continuous, namely:

$$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 \quad \forall x_1, x_2 \in [0,1], \ |x_1 - x_2| <$$

$$< \delta_\varepsilon \Rightarrow |x_2^2 - x_1^2| < \varepsilon$$

Considering the division Δ so that $||\Delta|| < \delta_\varepsilon$ (possibly because $||\Delta|| \to 0$), we have $x_i^2 - x_{i-1}^2 < \varepsilon$, so:

$$S_\Delta(f) - s_\Delta(f) = \sum_{i=1}^{n} \varepsilon(x_i - x_{i-1}) =$$

$$= \varepsilon \sum_{i=1}^{n} (x_i - x_{i-1}) = \varepsilon(b - a).$$

Because ε is arbitrarily small, we deduce that:

$$\lim_{\|\Delta\| \to 0} \left(S_\Delta(f) - s_\Delta(f) \right) = 0 ,$$

i.e. $f(x) = x^2$ is integrable on $[0,1]$.

References

T. ANDREESCU & collective: *Mathematical problems from competitions and exams sustained in 1983*, printed at the Polygraphic Enterprise “Banat”, Timisoara, 1984.

L. ARAMA, T. MOROZAN: *Problems of differential and integral calculus*, Ed. Tehnica, Bucharest, 1978.

C. AVADANEI, N. AVADANEI, C. BORS, C. CHIREA: *From elementary to superior mathematics*, Ed. Academiei R.S.R, Bucharest, 1977.

D. M. BATINETU: *Mathematical problems for the secondary high school*, Ed. Albatros, Bucharest, 1979.

D. M. BATINETU-GIURGIU, V. GHIORGHITA, I. V. MAFTEI, I. TOMESCU, F. VORNICESCU: *Problems given at mathematical Olympiads for high schools*, Ed. Stiintifica, Bucharest, 1992.

D. BUSNEAG, I. MAHTEI: *Homework for circles and mathematical contests for students*, Ed. Scrisul Romanesc, Craiova, 1983.

A. CORDUNEANU, G. RADU, L. POP, V. GRAMADA: *Mathematical problems workbook*, Ed. Junimea, Iasi, 1972.

GH. CALUGARITA, V. MANGU: *Mathematical problems for the elementary and secondary level in high school*, Ed. Albatros, Bucharest 1977.

M. COCUZ: *Mathematical problems workbook*, Ed. Academiei R.S.R, Bucharest, 1984.

N. DONCIU, D. FLONDOR: *Algebra and calculus, workbook with problems*, E.D.P., Bucharest, 1978.

M. DANGA: *Homework and mathematical problems*, Ed. Tehnica, Bucharest, 1991.

B. R. GELBAUM, J. M. H. OLMSTED: *Counterexamples in the analysis*, Ed. Stiintifica, Bucharest, 1973.

I. GIURGIU, F. TURTOIU: *Mathematical problems workbook*, E.D.P, Bucharest, 1980.

I. ILIESCU, B. IONESCU, D. RADU: *Mathematical problems for the admission in the higher education*, E.D.P., Bucharest, 1976.

C. IONESCU-TIU, L. PIRSAN: *Differential and integral calculus for the admission in college*, Ed. Albatros, Bucharest, 1975.

AL. LEONTE, C.P.NICULESCU: *Workbook of algebraic and calculus problems*, Ed. Scrisul Romanesc, Craiova, 1984.

I. A. MARON: *Problems in calculus of one variable*, Mir Publishers, Moscow, 1973.

A. S. MURESAN, V. MURESAN: *Algebraic and calculus problem, admission contests 1981-1990,* Ed. Tehnica, Bucharest, 1991.

C. NASTASESCU, C. NITA, M. BRANDIBURU, D.JOITA: *Algebraic exercises and problems*, E. D. P., Bucharest, 1981.

C. P. NICOLESCU: *Calculus*, Ed. Albatros, Bucharest, 1987.

C. P. NICOLESCU: *Mathematical synthesis*, Ed. Albatros, Bucharest, 1990.

G. RIZESCU, E. RIZESCU: *Homework for high school mathematical circles*, Ed. Tehnica, Bucharest, 1977.

V. SCHNEIDER, GH. A. SCHNEIDER: *Calculus problems workbook for grades XI-XII*, Ed. Hyperion-Apollo, Craiova, 1993.

GH. SIRETCHI: *Calculus problems*, Typography of the Bucharest University, 1975.

N. TEODORESCU, A. CONSTANTINESCU, M. MIHAI, L. PIRSAN, E. PERJERIU, A. POPESCU-ZORICA, P. RADOVICI-

MARCULESCU, M. TENA : *Problems from the mathematical gazette, methodological and selective edition*, Ed. Tehnica, Bucharest, 1984.

R. TRANDAFIR, AL. LEONTE: *Mathematical problems and exercises workbook*, Ed. Junimea, Iasi, 1975.

*** *Exercises workbook for the admission in the higher education, Mathematics, Physiscs and Chemistry*, Ed. St. Encicl., Bucharest, editions 1984 and 1989.

*** *High school manuals, different editions.*

*** *The Mathematical Gazette collection*.

In this book, we discuss a succession of methods encountered in the study of high school calculus to students and teachers, to higher education entry examination candidates, to all those interested, in order to allow them to reduce as many diverse problems as possible to already known work schemes.

We tried to present in a methodical manner, advantageous for the reader, the most frequent calculation methods encountered in the study of the calculus at this level.

ISBN 978-1-59973-355-5